A New Approach to Finite Element Simulations of General Relativity

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Abstract

In order to study gravitational waves, we introduce a new approach to finite element simulation of general relativity. This approach is based on approximating the Weyl curvature directly through new stable mixed finite elements for the Einstein-Bianchi system. We design and analyze these novel finite elements by adapting the recently developed Finite Element Exterior Calculus (FEEC) framework to abstract Hodge wave equations. This framework enables us to borrow key ideas from Reissner-Mindlin plate bending and elasticity with weakly imposed symmetries to maintain stability of the method. The stability of a discretization often relies on deep connections between fundamental branches of mathematics: the FEEC mimics these connections for the numerical method to achieve similar stability to that of the original equations. The recent development of FEEC has had a transformative impact on electromagnetism and related computational problems, and we are expanding it to general relativity.

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Chapter 1

Introduction

We introduce a new approach to finite element simulation of general relativity. We do so by designing and analyzing novel finite elements and adapting a recently developed framework to abstract wave equations and numerical relativity. The method is then implemented and applied to the study of gravitational waves.

Gravity is an universal attraction a massive object produces towards other massive objects. How masses and the gravity they induce change and bend space and time is described by Einstein's theory of general relativity. The equations describing gravity predict a new kind of wave that propagates in the universe. These gravitational waves can be understood as small ripples in the fabric of space-time itself caused by moving masses.

The objective of this thesis is to present the design and implementation of a new approach to computer simulation of the propagation of gravitational waves. Since Einstein's equations cannot be solved analytically except under special circumstances, numerical calculations are essential. In particular, they allow for inverting the signals from new observatories that sense gravity to uncover the nature of their sources, by producing expected signals to match with observed ones. However, computer simulations have themselves proven very challenging. These difficulties are in part due to the non-uniqueness of the solutions caused by the translation of the problem from geometric equations, relating lengths in the four-dimensional space-times, to differential equations, relating kinematics of objects in time.

The new computer simulation is based on an analogy between general relativity and electromagnetism: the *EB formulation of Einstein's equations*. The constrained evolution system is similar to Maxwell's equations with the essential difference of replacing unknown vectors by unknown symmetric and traceless matrices. The formulation is inspired by previous work [2, 3]. Because of the algebraic constraints, symmetry and traceless, imposed by the formulation, it turns out that this similarity with electromagnetism is not enough to obtain a convergent computer simulation. This implementation is the first computer implementation of the EB formulation, achieved by tailoring a new FEM to that formulation.

This thesis presents a way of maintaining stability while imposing the constraints by extending to this problem a recently developed general abstract framework, the Finite Element Exterior Calculus (FEEC) [4, 5]. Indeed, this stability property often relies on deep connections between fundamental branches of mathematics: the FEEC mimics these connections for the discretization to achieve similar stability to that of the original equations. This method is the first application of FEEC to general relativity.

1.1 Outline

In the first chapter, we introduce manifolds and curvature in the context of general relativity. We also review the standard $3 + 1$ linearization of Einstein's equations in order to identify small plane wave solutions.

In the second chapter, we introduce the Bel decomposition which divides the Weyl curvature tensor into its electric and magnetic part, following [2, 3]. We then derive a constrained evolution system for electric and magnetic parts of the Weyl tensor in the case of a small perturbation from Minkowski space: the linearized EB system. These two parts are symmetric, traceless, and divergence-free, matrices. We show how the evolution equations propagate some of those constraints, and review known results about this system.

In the third chapter, we develop a general framework enabling the study of wave equations such as Maxwell's equations and the linearized EB system. To do so, we introduce the theory of Hilbert complexes combined with the theory of unbounded operators needed in the general analysis of wave equations attached to complexes. We call these wave equations, Hodge wave equations. We then analyze the existence and uniqueness of a solution to a general Hodge wave equation. We then identify three formulations of the linearized EB system. The first one is based on the vector de Rham complex, and the abstract analysis carries over naturally. In this system, all the constraints are evolved by the evolution equations. However, it is not clear that the propagation of constraints is robust to adding coefficients and lower order terms. Therefore, we consider a version in which symmetries are imposed strongly. Finding finite elements for this formulation with strong symmetries is difficult. Indeed, this formulation requires symmetry and H^2 regularity. On one hand, as was discovered for elasticity, finite elements with symmetry are difficult to identify. On the other, as for plate bending, finite elements with H^2 regularity are also difficult to identify. To resolve the first issue, we present an alternative formulation with symmetry imposed weakly. To address the second issue, we introduce a new framework in the following chapter. This leads to a third formulation, but a different analysis needs to be provided for this formulation. This is the content of the next two chapters.

In the fourth chapter, we identify a formulation of the linearized EB system in which the symmetries are imposed weakly and the need for H^2 regularity is alleviated. We call this formulation the linearized EB system with weak symmetries. This formulation is identified by combining two complexes together. The general framework to do so is called the BGG framework. The name is due to the similarity of this process to the Bernstein-Gelfand-Gelfand resolution in the representation theory of Lie algebras. For this formulation, we show existence and uniqueness of a solution at the continuous level. We mimic this analysis at the discrete level to provide new mixed finite elements for this problem. The general analysis of such combination of two complexes finds inspiration in elasticity with weak symmetry and a limiting case of Reissner–Mindlin plate bending. As an application of this general framework and analysis, we consider the time-independent linearized EB system with weak symmetries. We then extend the framework to the time-dependent case. This enables us to analyze the time-dependent linearized EB system with weak symmetries. We show the convergence analysis of the method.

In the fifth chapter, we investigate the implementation in the case of small perturbations and of gravitational waves. Due to the presence of projection operators in the linearized EB system, we hybridize the system by introducing multipliers and integrals on edges and faces to ease the implementation. Moreover, to improve efficiency of the method, we derive a preconditioner tailored to this hybridized system. For the implementation, we then use FEniCS, an open source project with focus on solving PDE by finite element methods. With this implementation, we confirm the convergence of the method. We also demonstrate the propagation of a gravitational wave from a weak source using the method.

Chapter 2

Manifolds, Tensors, and General Relativity

In general relativity, four-dimensional spacetime is represented by a smooth manifold endowed with a metric. The geodesics on this manifold describe the paths of particles in free fall. In this chapter, we introduce background material on manifold geometry that is needed to discuss Einstein's equations. For this, we follow [6], and also refer to [1, 7, 8, 9]. We begin by tensor fields on manifolds and the abstract index notation to compute with them. This then leads us to the metric and associated covariant derivative. Following this, we discuss the curvature of a smooth manifold. Einstein's equations then tell us the relation between a given distribution of matter and the curvature of the manifold. Finally, we discuss the linearization of Einstein's equations with the goal of understanding plane wave solutions.

2.1 Tensor Fields on Manifolds

We begin by recalling the definition of a manifold. An n-dimensional (smooth and real) manifold is a set M equipped with a collection of subsets $\{O_\alpha\}$ such that

- $\{O_\alpha\}$ covers M;
- for each α, there exists a bijective map $\psi_\alpha : O_\alpha \to U_\alpha$, where U_α is an open subset of $\mathbb{R}^n$;
- for any α and β such that $O_\alpha \cap O_\beta$ is not empty, $\psi_\beta \circ \psi_\alpha^{-1}$ is a smooth map between

the $\psi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha \subset \mathbb{R}^n$ to $\psi_\beta(O_\alpha \cap O_\beta) \subset U_\beta \subset \mathbb{R}^n$, assumed open in $\mathbb{R}^n$.

The maps ψ_α is said to be a *chart* or a *coordinate system*. We always assume that we have a *maximal* cover $\{O_\alpha\}$ and chart family: all charts compatible with the last two properties are included.

For any p in an n-dimensional manifold M, we can attach an n-dimensional vector space T_pM, called the *tangent space* of M at p. An element $\boldsymbol{v}$ of T_pM is a *tangent vector* at p: a map on smooth scalar functions on M to $\mathbb{R}$ such that

- $\boldsymbol{v}$ is linear,
- $\boldsymbol{v}$ satisfies the Leibniz Rule: $\boldsymbol{v}(fg) = \boldsymbol{v}(p)v(g) + \boldsymbol{v}(f)g(p)$ for any smooth scalar functions f, g on M.

The *cotangent space* at p is the dual space T_p^*M of the tangent space T_pM. A covector is thus a linear functional on T_pM. As T_pM is finite dimensional, its dual T_p^*M has the same dimension.

At any $p \in M$, we can define the *coordinate basis* of T_pM for a chart ψ around p,

$$X_\mu(f) = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\bigg|_{\psi(p)}$$

where x^μ are the Cartesian coordinates of $\mathbb{R}^n$, μ is an integer with $1 \leq \mu \leq n$, and f is a smooth map from M to $\mathbb{R}$. If we pick another chart ψ' around p, the chain rule gives us the relation between X_μ and X'_μ,

$$X_\mu = \sum_\nu \frac{\partial x'^\nu}{\partial x^\mu}\bigg|_{\psi(p)} X'_\nu,$$

where x'^ν denotes the νth component of $\psi' \circ \psi^{-1}$. We often denote X_μ by $\partial/\partial x^\mu$ or simply by ∂_μ. The associated dual basis for T_p^*M is given by $dx^1(p), \ldots, dx^n(p)$: the notation is just a symbol for the linear map such that $dx^\mu\left(\partial_\nu\right) = \delta^\mu{}_\nu$ at p. At the point p, we can expand a vector $\boldsymbol{v}$ in terms of its *components* v^μ in the given basis: $\boldsymbol{v} = \sum_{\mu=1}^n v^\mu \partial_\mu = v^\mu \partial_\mu$, where we used *Einstein's summation convention* which imply the summation over repeated Greek indices. Similarly, a covector $\boldsymbol{w}$ at p can be written in terms of its components w_μ as $\boldsymbol{w} = \sum_{\mu=1}^n w_\mu dx^\mu = w_\mu dx^\mu$.

For any non-negative integers k and l and for any point $p \in M$, we consider the tensor product

$$T_p^{k,l}M := \underbrace{T_pM \otimes \cdots \otimes T_pM}_{k} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{l}$$

which is a vector space of dimension n^{k+l}. This tensor product can be identified with the space of (k, l)-linear maps

$$\underbrace{T_p^*M \times \cdots \times T_p^*M}_{k} \times \underbrace{T_pM \times \cdots \times T_pM}_{l} \to \mathbb{R}.$$

An element of $T_p^{k,l}M$ is called a tensor at p, and the pair (k, l) is the *valence* of the tensor. In particular, a tangent vector at p is an element of $T_p^{1,0}M$, and a covector is an element of $T_p^{0,1}M$. A tensor of arbitrary valence can be expanded in terms of its components in the basis and dual basis.

The disjoint union of the tangent spaces T_pM is the *tangent bundle* TM, and may be given the structure of a $2n$-dimensional manifold. Similarly, we can construct the *cotangent bundle* T^*M, and the (k, l)-tensor tangent bundle $T^{k,l}M$. A *vector field* on M, that is a section of TM, is a smooth function $v : M \to TM$ such that $v(p) \in T_pM$. Similarly, we can construct *covector fields* and (k, l)*-tensor fields*. A (k, l)-tensor field is thus a function mapping $p \in M$ to a (k, l)-multilinear map on k covectors and l vectors. The set of all (k, l)-tensor fields is $\Gamma(T^{k,l}M)$. We highlight that the set of $(0, 0)$-tensor fields are scalar fields, of $(1, 0)$-tensor fields are vector fields, and $(0, 1)$-tensor fields are covector fields.

We use two notations for tensor fields (and for tensors when mentioned explicitly in the context). First, we use bold characters to denote tensors of any valence. This notation will also be convenient when discussing differential equations, combined with standard differential operators like grad, curl, and div. Second, we use the *abstract index notation* which is often more convenient for computation. The notation is the following. For a vector, we may use v^a – the choice of a for the superscript has no particular meaning other than indicating that the object is a vector. In this notation, (k, l)-tensor fields are denoted with k distinct superscripts and l distinct subscripts. We call the superscripts *contravariant indices* and the subscripts *covariant indices*. To refer to a tensor field of arbitrary valence, we may write $v^{a\cdots}{}_{c\ldots}$. In order to avoid confusing expressions written in the abstract index notation and expressions written in terms of components in a coordinate system, we employ the following convention inspired by [6]: we use Latin indices starting at a for abstract indices, and Greek indices for tensors in a given coordinate system.

The tensor product of two tensor fields, say v^a and w_b is written, in abstract index notation, simply as $v^a w_b$. This is inspired by the coordinate expression, since, if v^μ and w_ν give the components of two tensor fields with respect to the same coordinate system, then the components of their tensor product is indeed $v^\mu w_\nu$. This notation readily extends to the tensor product between tensors of arbitrary valence. It is important to make the distinction between $v^a w_b$ denoting the tensor product of v^a and w_b, and $v^a w_a$. The latter denotes the *contraction* of v^a with w_b and its result is the scalar field $v^a w_a$ such that, at any $p \in M$, $(v^a w_a)(p)$ is equal to the real number obtained when $v^a(p)$ is paired with $w_b(p)$. The contraction is indicated by the repetition of the index. In general, the contraction of the ith and the jth indices is a map $\Gamma(T^{k,l}M) \to \Gamma(T^{k-1,l-1}M)$, by pairing the ith copy of T^*_pM with the jth copy of T_pM. We call the (i,j)th *trace* of a tensor field the contraction of its ith contravariant index and jth covariant index. For instance, the $(1,1)$-trace on $T^{1,1}_pM$ is the map $T^{1,1}_pM \to \mathbb{R}$ sending a tensor $h^b{}_c$ to $h^b{}_b$.

In order to respect the vector space structure of tensors, writing equations in abstract index notation has to follow the following rule. In an equation, each term's superscripts/subscripts should have the same number of any non-contracted letter, so that contracting with a fixed covector/vector removes a letter from each term. For instance, we can write $a^{abcd}{}_{ef} + b^{abdc}{}_{ef} = h^{abdc}{}_{ef}$. For a $(2,0)$-tensor field h^{ab}, we can write $h^{ab} + h^{ab} = 2h^{ab}$ for an equation in $T^{2,0}_pM$. However, this is in general not the same $(2,0)$-tensor field as $h^{ab} + h^{ba}$, since $h^{ab}v_a w_b$ does not result in the same scalar field as $h^{ba}v_a w_b$. Since $T^{2,0}_pM = T_pM \otimes T_pM$, we can see $h^{ab} \mapsto h^{ba}$ as the map $\boldsymbol{u} \otimes \boldsymbol{v} \mapsto \boldsymbol{v} \otimes \boldsymbol{u}$ for vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$. We introduce the following elements of notation: the use of parenthesis for the symmetric part and square-brackets for the anti-symmetric part of tensor fields. For instance, $v^{a(bc)} = \frac{1}{2}(v^{abc} + v^{acb})$ and $v^{a[bc]} = \frac{1}{2}(v^{abc} - v^{acb})$. Thus, we have that $v^{a(bc)} = v^{a(cb)}$ and $v^{a[bc]} = -v^{a[cb]}$, as desired. For a tensor field of any valence, the symmetric part is the average of all the permutations of the indices with respect to which one is symmetrizing. The anti-symmetric part is the same sum with each term multiplied by the sign of the permutation. For example, $v_{a[bcd]} = \frac{1}{6}\left(v_{abcd} + v_{adbc} + v_{acdb} - v_{abdc} - v_{acbd} - v_{adcb}\right)$.

2.2 Metric

A *pseudo-Riemannian* metric on M is a non-degenerate symmetric $(0,2)$-tensor field g_{ab}. *Non-degenerate* means that, for any point $p \in M$ and nonzero vector $v^a \in T_pM$, there exists a vector w^b such that $g_{ab}(p)v^a w^b$ is not zero. Since $g_{ab}(p)$ is a symmetric bilinear form for every $p \in T_pM$, there is a basis of vectors ${}^1v^a, \ldots, {}^nv^a$ diagonalizing $g_{ab}(p)$ at p, namely that $(g_{ab}(p))({}^iv^a)({}^jv^b) = 0$ if $i \neq j$. Moreover, the scalar $\lambda_i := (g_{ab}(p))({}^iv^a)({}^iv^b)$ is not zero for any i. By Sylvester's law of inertia, the number of λ_i which are positive does not depend on the choice of basis. This number determine the *signature* of the metric [9, p. 42]. If this number is n, then $g_{ab}(p)$ is positive definite for each T_pM and thus induces an inner product on T_pM. The metric is then called *Riemannian*. In contrast, if this number is $n-1$, then the metric only induces a pseudo-inner product (symmetric and non-degenerate, but not positive definite) on the tangent spaces. The metric is then called *Lorentzian*. We sometimes denote the signature as a list of $+$ and $-$. For instance, we might say that the metric has signature $(-+++)$ for a 4-dimensional manifold equipped with a Lorentzian metric. For a Lorentzian metric g_{ab}, a vector v^a at p is called spacelike if $g_{ab}(p)v^a v^b > 0$, timelike if $g_{ab}(p)v^a v^b < 0$, and lightlike if $g_{ab}(p)v^a v^b = 0$.

The metric induces a map between vector fields and covector fields, as it induces a map $g_{ab}(p) : T_pM \to T_p^*M$ at any point p of the manifold. Since the metric is non-degenerate, this map is invertible at any $p \in M$. We denote the inverse by $g^{ab}(p)$ and the associated tensor field g^{ab}. Contracting the metric with its inverse g^{bc} gives $g_{ab}g^{bc} = \delta_a{}^c = \delta^c{}_a$, the identity $(1,1)$-tensor field, called the *Kronecker delta*. Using the metric and its inverse, we can *lower* the index of a vector field and *raise* the index of a covector field. Indeed, for a given vector field v^a, we define the covector field $v_a = g_{ab}v^b$ with lowered index. Similarly, for a covector field w_a, we define the vector field $w^a = g^{ab}v_b$ with raised index. We can also raise and lower indices of any tensor fields. Since g^{ab} is the inverse of g_{ab}, raising and lowering indices are inverse of each other: $g_{ab}g^{bc}v_c = \delta_a{}^c v_c = v_a$. By raising and lowering indices, we may end up with staggered indices on a tensor field in the abstract index notation. For instance, for the $(3,2)$-tensor field $h^{abc}{}_{de}$, we have $h_{ab}{}^c{}_d{}^e = g_{af}g_{bg}g^{eh}h^{fgc}{}_{dh}$. To refer to a tensor field of arbitrary valence, we still write $v^{a\cdots}{}_{c\ldots}$ in the abstract index notation. Moreover, since we can change the valence of a tensor field using the metric, we sometimes refer to (k, l)-tensor fields as $(k + l)$-tensor fields.

A metric on M uniquely determines a *covariant derivative* ∇_a as follows [6, p. 31]. The covariant derivative ∇_a (also denoted $\boldsymbol{\nabla}$) takes a (k, l)-tensor field, for any non-negative integers k and l, and returns a $(k, l + 1)$-tensor field such that

- Linearity: ∇_a is linear from $\mathcal{T}^{k,l}M$ to $\mathcal{T}^{k,l+1}M$.
- Leibniz Rule: ∇_a satisfies $\nabla_a(v^{b\dots}{}_{c\dots} w^{d\dots}{}_{e\dots}) = (\nabla_a v^{b\dots}{}_{c\dots})(w^{d\dots}{}_{e\dots}) + (v^{b\dots}{}_{c\dots})(\nabla_a w^{d\dots}{}_{e\dots})$.
- Commutativity with traces: ∇_a commutes with contractions of indices, $\nabla_a(v^{\dots b\dots}{}_{\dots b\dots}) = \nabla_a v^{\dots b\dots}{}_{\dots b\dots}$.
- Differential on scalars: ∇_a is the usual exterior derivative on scalar fields, $(\nabla_a f)(p) = df_p$ for any $p \in M$ and $f \in \mathcal{T}^{0,0}M$.
- Torsion-free: ∇_a commutes on scalar fields, $\nabla_a \nabla_b f$ is symmetric for any $f \in \mathcal{T}^{0,0}M$.
- Compatibility with the metric: $\nabla_a g_{bc} = 0$.

We also sometimes denote the covariant derivative with a semi-colon: for instance, $v_{a;b} := \nabla_b v_a$ for a covector v_b.

In a given coordinate system with the coordinate basis x^μ, the covariant derivative can be computed explicitly. From the definition, we can already compute it for scalars,

$$\nabla_\mu f = f_{,\mu}$$

where we define $f_{,\mu} = \partial f / \partial x^\mu$. The coordinate expression of the covariant derivative of a tensor of arbitrary valence requires *Christoffel symbols*. The Christoffel symbols are given as

$$\Gamma^\mu_{\nu\alpha} := g^{\mu\gamma} \Gamma_{\nu\alpha\gamma} := \frac{1}{2} g^{\mu\gamma} \left(g_{\nu\gamma,\alpha} + g_{\alpha\gamma,\nu} - g_{\nu\alpha,\gamma} \right). \tag{2.1}$$

Since the metric is symmetric, we have that $\Gamma^\mu_{\nu\alpha} = \Gamma^\mu_{\alpha\nu}$. The Christoffel symbols are always attached to a coordinate system. Now, we can compute the covariant derivative of vectors and covectors:

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\alpha} v^\alpha \text{ and } \nabla_\mu v_\nu = \frac{\partial v_\nu}{\partial x^\mu} - \Gamma^\alpha_{\mu\nu} v_\alpha.$$

Higher-order tensors repeat the pattern of vectors and covectors. For instance,

$$\nabla_\mu v^\nu{}_\gamma = \frac{\partial v^\nu{}_\gamma}{\partial x^\mu} + \Gamma^\nu_{\mu\alpha} v^\alpha{}_\gamma - \Gamma^\alpha_{\mu\gamma} v^\nu{}_\alpha.$$

2.3 Riemann Curvature Tensor

Even though the covariant derivative commutes on scalar fields, it does not commute on arbitrary tensor fields. For example, the quantity $\nabla_{[c}\nabla_{d]}v^a$ need not vanish for a vector field v^a. However, it is linear in v^a at every point $p \in M$, and can be shown to have its value at a point $p \in M$ only dependent on the value of v^a at p [9, p. 33]. We then define the *Riemann curvature tensor* as the tensor field $R^a{}_{bcd}v^b := \nabla_{[c}\nabla_{d]}v^a$ following [1]. We often simply say Riemann tensor.

Using the explicit form of the covariant derivative in a given coordinate basis, we give an explicit way of computing the Riemann tensor from the metric. If we introduce a system of coordinates, the components of the Riemann tensor are given by

$$R^\alpha{}_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\gamma\beta}\Gamma^\gamma_{\mu\nu} - \Gamma^\alpha_{\gamma\nu}\Gamma^\gamma_{\mu\beta}. \tag{2.2}$$

A vector field V^a is said to be *parallel transported* along a curve with tangent field T^a when $T^b\nabla_b V^a = 0$. In particular, a *geodesic* is a curve whose tangent vector field T^a is parallel transported along itself: $T^b\nabla_b T^a = 0$. The Riemann curvature tensor then describes whether geodesics on a manifolds become closer or further away from each other. Indeed, consider a smooth one-parameter family of geodesics $\gamma_s(t)$. Then we can use t and s as coordinates for the 2-dimensional submanifold spanned by the geodesics, and set $T^a = (\partial/\partial t)^a$ and $X^a = (\partial/\partial s)^a$. The vector field $a^a = T^c\nabla_c(T^b\nabla_b X^a)$ tells us the relative acceleration of nearby geodesics. A calculation as done in [1, p. 37] and [6, p. 47], using the definition of the Riemann tensor, the Leibniz rule, and the fact that the coordinate vector fields T^a and X^b commute as operators on scalar fields, yields the *geodesic deviation equation*

$$a^a = -R^a{}_{cbd}X^bT^cT^d \tag{2.3}$$

which then tells us that initially parallel geodesic will fail to remain so if and only if the Riemann curvature tensor does not vanish. Thus, parallel straight lines remain parallel on a flat manifold.

Even though the Riemann tensor has four indices and could have up to n^4 independent components, it really only has $n^2(n^2-1)/12$ independent components (20 when $n = 4$) because of its many symmetries.

Proposition 2.1. *The Riemann tensor satisfies*

- *skew-symmetry in each of the two pairs* $R_{abcd} = -R_{bacd} = -R_{abdc}$*,*
- *the first (or algebraic) Bianchi identity,* $R_{a[bcd]} = 0$*.*

Proof. The skew-symmetry in the first pair follows from the compatibility of the metric:

$$0 = 2\left(\nabla_c \nabla_d - \nabla_d \nabla_c\right) g_{ae} = R_a{}^b{}_{cd} g_{be} + R_e{}^b{}_{cd} g_{ab} = R_{aecd} + R_{eacd}.$$

The skew-symmetry in the second pair is clear from the definition. The first Bianchi identity follows by observing that $0 = \nabla_{[a} \nabla_b v_{c]}$ by expanding the covariant derivative in any coordinate basis (this relation in differential forms is simply that $d^2 \vec{v} = 0$ for a vector $\vec{v}$ and exterior derivative d). From this relation, we see immediately that

$$0 = \nabla_{[c} \nabla_d v_{a]} - \nabla_{[d} \nabla_c v_{a]} = R_{[a}{}^b{}_{cd]} v_b,$$

as desired. □

We can write the symmetries in the previous proposition differently. From [10], we have the following for any tensor field C_{abcd} satisfying skew symmetry in each pair: C_{abcd} satisfies the first Bianchi identity if and only if C_{abcd} has the interchange symmetry, $C_{abcd} = C_{cdab}$, and null totally antisymmetric part, $C_{[abcd]} = 0$. Finally, another important property of the Riemann tensor is the (second or differential) Bianchi identity,

$$\nabla_{[e} R_{cd]ab} = 0, \tag{2.4}$$

which can be obtained from the definition.

The trace of the Riemann tensor is called the *Ricci tensor*, $R_{ab} = R^c{}_{acb}$, and is a $(0, 2)$-tensor field. Because of the interchange symmetry of the Riemann tensor, the Ricci tensor is symmetric. If the Ricci tensor vanishes identically, we say that the manifold is *Ricci-flat*. The trace of the Ricci tensor is the *Ricci scalar*, or *scalar curvature*, $R = R^a{}_a$, and is a scalar field. Now, we take the trace twice (e with a, and b with d) in the second Bianchi identity (2.4), and get the twice-contracted Bianchi identity, $\nabla^a \left(R_{ac} - \frac{1}{2} g_{ac} R\right) = 0$. The *Einstein tensor*

$$G_{ab} = R_{ac} - \frac{1}{2} g_{ac} R$$

is the $(0, 2)$-tensor field appearing in parenthesis, and inherits symmetry from the Ricci tensor. The Einstein tensor is not an arbitrary symmetric tensor though, as we have just shown it satisfies

$$\nabla^a G_{ac} = 0.$$

2.4 Einstein's Equations

In the theory of general relativity, the four-dimensional spacetime is represented by a four-dimensional Lorentzian manifold equipped with a metric g_{ab} of signature $(-+++)$. The relation between the curvature of the manifold and the mass and energy living on it is the content of *Einstein's equations* (sometimes also called *Einstein's field equations*). These equations are a set of 10 equations that relate the Einstein tensor G_{ab} and the energy-momentum tensor T_{ab} through

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab},$$

where G on the right is the gravitational constant and c is the speed of light. The constant $8\pi G/c^4$ then has a value of about $5 \cdot 10^{-44} \mathrm{s}^2/\mathrm{m}/\mathrm{kg}$, and thus highlights the scale of mass and energy needed to create a noticeable effect. For convenience, we will take *geometrized units* in which $G = c = 1$, so that the constant is simply 8π.

Since the twice-contracted Bianchi identity says that the Einstein tensor satisfies the conservation law $\nabla^a G_{ab} = 0$, the energy-momentum tensor should be *conservative*, namely that $\nabla^a T_{ab} = 0$. In the case of vacuum, $T_{ab} = 0$, so $G_{ab} = 0$. Since the trace of G_{ab} is zero if and only if the trace of R_{ac} is zero, $G_{ab} = 0$ is equivalent to $R_{ab} = 0$. Thus, in vacuum, Einstein's equations state that the manifold is Ricci-flat.

2.5 Gauge Freedom

If we are given two manifolds M and N of dimensions m and n with charts ψ_α^M and ψ_β^N, a map $\phi : M \to N$ is said to be *smooth*, if, for each α and β, $\psi_\beta^N \circ \phi \circ (\psi_\alpha^M)^{-1}$ mapping $U_\alpha \subset \mathbb{R}^m$ into $U_\beta \subset \mathbb{R}^n$ is smooth. If also ϕ is bijective and has a smooth inverse, then ϕ is a *diffeomorphism*. In that case, M and N are said to be *diffeomorphic* and have identical manifold structures [6].

If M and N are two manifolds and $\phi : M \to N$ is a smooth map, then, at each point $p \in M$, the *differential* $d\phi_p : T_pM \to T_{\phi(p)}N$ is a linear map such that $d\phi_p(\boldsymbol{X})(f) = \boldsymbol{X}(f \circ \phi)$ for $\boldsymbol{X} \in T_pM$ and f a smooth scalar field on N. We can then define the *push-forward* of vectors, $\phi_* : TM \to TN$ and $\boldsymbol{v} \in T_pM$, $(\phi_*\boldsymbol{v})_p := d\phi_p\boldsymbol{v} \in T_{\phi(p)}N$. The adjoint map $(d\phi_p)^*$ maps the other way: from $T^*_{\phi(p)}N$ to T^*_pM. To combine these adjoint maps at every $p \in M$, ϕ needs to be one-to-one and onto. Thus, if ϕ is a diffeomorphism, then the *pull-back* ϕ^* from T^*N to T^*M, can be defined as $(\phi^*)_p\boldsymbol{w} := (d\phi_p)^*\boldsymbol{w} \in T^*_pM$ for any $p \in M$ and $\boldsymbol{w} \in T^*_{\phi(p)}N$. In the case when ϕ is a diffeomorphism, and thus invertible, we can also define the push-forward of covectors via $\phi_* := (\phi^{-1})^* : T^*M \to T^*N$. Using the tensor products, we thus have a push-forward for tensors of any valence: $\phi_* : T^{k,l}M \to T^{k,l}N$. Similarly, we have a pull-back ϕ^* for tensors of any valence. We can also extend the definition of push-forward ϕ_* to tensor fields too: for the (k,l)-tensor field on M, we view $\phi_*\boldsymbol{v}$ as the (k,l)-tensor field on N mapping $\phi(p)$ to $\phi_*\boldsymbol{v}(p)$. Similarly, we also extend the definition of pull-back ϕ^*.

Suppose we are given two diffeomorphic manifolds M and N via the diffeomorphism ϕ, where M has metric $\boldsymbol{g}$. Then, the push-forward $\phi_*\boldsymbol{g}$ to N assigns the same value to a pair of vectors at $\phi(p) \in N$ as $\boldsymbol{g}$ as the corresponding pair of vectors at $p \in M$. Thus, by construction, $\phi_*\boldsymbol{g}$ on N is isometric to $\boldsymbol{g}$ on M. For an arbitrary tensor field $\boldsymbol{v}$, we get from the definitions that $\phi_*(\boldsymbol{\nabla}\boldsymbol{v}) = \boldsymbol{\nabla}(\phi_*\boldsymbol{v})$, where on the left $\boldsymbol{\nabla}$ is the covariant derivative attached to M and to N on the right with appropriate metrics. It follows that the Riemann tensor associated to $\phi_*\boldsymbol{g}$ on N is just the push-forward of the one on M, and similarly for the Ricci tensor, Ricci scalar, and Einstein tensor.

A solution to Einstein's equations then corresponds to an equivalence class of manifolds with a high degree of non-uniqueness. The equivalence relation is given by diffeomorphisms. This non-uniqueness cannot be suppressed with boundary conditions: indeed, if a manifold has a boundary, one can take a diffeomorphism reducing to the identity near the boundary. Therefore, whenever Einstein's equations has a solution, it actually has many. We refer to this non-uniqueness as *gauge freedom*, and call *gauge-related* a tensor field and its push-forward to another manifold of the same equivalence class. For instance, consider a metric $\boldsymbol{g}$ satisfying the vacuum Einstein's equations, $\boldsymbol{G} = 0$: for any diffeomorphism $\phi : M \to N$, the push-forward $\phi_*\boldsymbol{g}$ still satisfy the vacuum Einstein's equations. Often, it is necessary to pick a single representative in the equivalence class of solutions, and, to do so, we can impose some extra conditions, called *gauge conditions*. This process of picking an unique representative is called *gauge fixing*.

2.6 Lie Derivative

When linearizing in the next section, we consider one-parameter groups of diffeomorphisms on a background manifold. Such one-parameter groups of diffeomorphisms $\phi : M \times \mathbb{R} \to M$ on a manifold M are in fact induced by vector fields. Thus, for a vector field $\boldsymbol{Y}$ and any tensor field $\boldsymbol{T}$, the *Lie derivative* at $p \in M$ is then

$$(\mathcal{L}_{\boldsymbol{Y}}\boldsymbol{T})(p) = \left.\frac{d}{dt}\right|_{t=0} ((\phi_*)_{-t}T)(p)$$

where ϕ_t^* is the pullback associated to the diffeomorphism ϕ_t induced by $\boldsymbol{Y}$ (see [9, p. 20] [6, p. 439]). The Lie derivative satisfies Leibniz rule: for any tensor fields $\boldsymbol{S}$ and $\boldsymbol{T}$, $\mathcal{L}_{\boldsymbol{Y}}(\boldsymbol{S} \otimes \boldsymbol{T}) = (\mathcal{L}_{\boldsymbol{Y}}\boldsymbol{S}) \otimes \boldsymbol{T} + \boldsymbol{S} \otimes (\mathcal{L}_{\boldsymbol{Y}}\boldsymbol{T})$. For a scalar field f, the Lie derivative becomes $\mathcal{L}_{\boldsymbol{Y}} f = Y(f)$. For a vector field $\boldsymbol{X}$, the Lie derivative is $\mathcal{L}_{\boldsymbol{Y}}\boldsymbol{X} = [\boldsymbol{Y}, \boldsymbol{X}]$, where $[\boldsymbol{Y}, \boldsymbol{X}]$ is the *Lie bracket* or *commutator* of $\boldsymbol{X}$ and $\boldsymbol{Y}$ and $[\boldsymbol{Y}, \boldsymbol{X}]^b = Y^a \nabla_a X^b - X^a \nabla_a Y^b$. Thus, for a vector field X^a, we have

$$\mathcal{L}_{\boldsymbol{Y}} X^b = Y^a \nabla_a X^b - X^a \nabla_a Y^b.$$

For a covector field X_a, we have

$$\mathcal{L}_{\boldsymbol{Y}} X_b = Y^a \nabla_a X_b + X_a \nabla_b Y^a.$$

For a tensor field with arbitrary valence, the pattern repeats: one extra term for each lower/upper index. In particular, if we apply the Lie derivative to the metric, we get $\mathcal{L}_{\boldsymbol{X}} g_{ab} = X^c \nabla_c g_{ab} + g_{cb} \nabla_a X^c + g_{ac} \nabla_b X^c = 2\nabla_{(a} X_{b)}$.

2.7 Linearization

We now review the case of small perturbations of the flat Minkowski metric in order to study the structure of Einstein's equations. Such small perturbations enable us to understand gravitational waves.

We follow [6, pages 75, 183, 441], [8, p. 277], and [9, p. 24], and we fix the background manifold $M_b = \mathbb{R}^4$ equipped with the Minkowski metric $\boldsymbol{\eta}$. We suppose that we have a smooth one-parameter family $\boldsymbol{g}(\epsilon)$ of exact solutions to Einstein's equations $\boldsymbol{G}(\epsilon) = 8\pi\boldsymbol{T}(\epsilon)$ such that $\boldsymbol{g}(0) = \boldsymbol{\eta}$ and $\boldsymbol{T}(0) = 0$. Thus, we can expand the family in a series, $\boldsymbol{g}(\epsilon) = \boldsymbol{\eta} + \epsilon\boldsymbol{h} + \mathcal{O}(\epsilon^2)$, where

$$\boldsymbol{h} = \frac{d}{d\epsilon}\boldsymbol{g}(\epsilon)\bigg|_{\epsilon=0}$$

is understood as a *perturbation* of $\boldsymbol{\eta}$. For simplicity, we also often call $\boldsymbol{h}$ the metric.

As usual, indices are raised and lowered using the Minkowski metric $\boldsymbol{\eta}$, except in the case of the metric for which g^{ab} denotes the inverse of g_{ab}, $g^{ab} \approx \eta^{ab} - \epsilon h^{ab}$, where $\approx$ means equality up to $\mathcal{O}(\epsilon^2)$. Moreover, the derivative operators are associated to the Minkowski metric.

We now want to know how perturbations of the metric change under diffeomorphisms.

Theorem 2.2. *Suppose we are given any vector field $\boldsymbol{\xi}$ on M_b. If we call the induced one-parameter group of diffeomorphisms $\psi(\epsilon) : M_b \to M_b$, then*

$$\frac{d}{d\epsilon}\psi^*(\epsilon)\boldsymbol{g}(\epsilon)\bigg|_{\epsilon=0} = \boldsymbol{h} + 2\mathcal{E}(\boldsymbol{\xi}).$$

We say that $\boldsymbol{h}$ and $\boldsymbol{h} + 2\mathcal{E}(\boldsymbol{\xi})$ are gauge-related*.*

Proof. Consider a vector field $\boldsymbol{\xi}$ on M_b. This vector field induces a flow, and thus also a one-parameter group of diffeomorphisms $\psi(\epsilon)$ on M_b. We thus study how the first order perturbation $\boldsymbol{h}$ changes to

$$\boldsymbol{h}' = \frac{d}{d\epsilon}\psi^*(\epsilon)\boldsymbol{g}(\epsilon)\bigg|_{\epsilon=0}.$$

Hence,

$$\psi^*(\epsilon)\boldsymbol{g}(\epsilon) = \psi^*(\epsilon)\left(\boldsymbol{\eta} + \epsilon\boldsymbol{h} + \mathcal{O}(\epsilon^2)\right) \approx \psi^*(\epsilon)\boldsymbol{\eta} + \epsilon\psi^*(\epsilon)\boldsymbol{h}.$$

We then further see that

$$\psi^*(\epsilon)\boldsymbol{h} = \boldsymbol{h} + \mathcal{O}(\epsilon),$$

and that

$$\psi^*(\epsilon)\boldsymbol{\eta} = \boldsymbol{\eta} + \epsilon\frac{\psi^*(\epsilon)\boldsymbol{\eta} - \boldsymbol{\eta}}{\epsilon} = \boldsymbol{\eta} + \epsilon\mathcal{L}_\xi(\boldsymbol{\eta}) + \mathcal{O}(\epsilon^2).$$

Therefore,

$$\psi^*(\epsilon)\boldsymbol{g}(\epsilon) \approx \boldsymbol{\eta} + \epsilon\boldsymbol{h} + \epsilon\mathcal{L}_{\boldsymbol{\xi}}(\boldsymbol{\eta}),$$

so $\boldsymbol{h}' = \boldsymbol{h} + \mathcal{L}_{\boldsymbol{\xi}}(\boldsymbol{\eta})$. However, we recall that $\mathcal{L}_{\boldsymbol{\xi}}(\eta_{ab}) = \nabla_a \xi_b + \nabla_b \xi_a$, which is simply the symmetric gradient, $2\mathcal{E}(\boldsymbol{\xi})$, as desired. □

This theorem describes the effect of a diffeomorphism on the perturbations of metrics. We thus have that gauge-related metrics are given by

$$h^{\text{new}}_{ab} = h_{ab} + 2\nabla_{(a}\xi_{b)},$$

for any covector field ξ_b.

To later simplify the equations, we introduce the trace-reversed metric, $\bar{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{\eta} h/2$, so that $\boldsymbol{g} \approx \boldsymbol{\eta} + \epsilon\bar{\boldsymbol{h}} - \epsilon\boldsymbol{\eta}\bar{h}/2$, where $h = \eta^{ab}h_{ab}$ and $\bar{h} = \eta^{ab}\bar{h}_{ab}$. A calculation using Theorem 2.2 shows that gauge-related trace-reversed metrics are given by

$$\bar{h}^{\text{new}}_{ab} = \bar{h}_{ab} + 2\nabla_{(a}\xi_{b)} - \eta_{ab}\nabla^c\xi_c, \tag{2.5}$$

since $\bar{h}^{\text{new}}_{ab} = h^{\text{new}}_{ab} - \eta_{ab}h^{\text{new}}/2$ and $h^{\text{new}} = h + 2\nabla^c\xi_c$.

We are now ready to see the effect of a diffeormorhism on the Einstein tensor.

Theorem 2.3. *The linearization* $\boldsymbol{H} = \frac{d}{d\epsilon}\boldsymbol{G}\Big|_{\epsilon=0}$ *of the Einstein tensor* $\boldsymbol{G}$ *is*

$$H_{bd} = \nabla^c\nabla_{(b}h_{d)c} - \frac{1}{2}\left(\nabla^c\nabla_c h_{bd} + \nabla_d\nabla_b h\right) - \frac{\eta_{bd}}{2}\left(\nabla^c\nabla^e h_{ec} - \nabla^c\nabla_c h\right)$$

or in terms of the trace-reversed tensor

$$H_{bd} = \nabla^a\nabla_{(b}\bar{h}_{d)a} - \frac{1}{2}\left(\nabla^a\nabla_a\bar{h}_{bd} + \eta_{bd}\nabla^a\nabla^e\bar{h}_{ea}\right).$$

Proof. We follow [1, p. 39]. We first compute the Christoffel symbol given in equations (2.1), the connection tensor relating the derivative operators associated to the metric $\boldsymbol{g}$ and $\boldsymbol{\eta}$. We denote the latter $\boldsymbol{\nabla}$. Thus,

$$\Gamma_{abc} = \frac{1}{2}\left(\nabla_b g_{ac} + \nabla_a g_{bc} - \nabla_c g_{ab}\right) \approx \epsilon\frac{1}{2}\left(\nabla_b h_{ac} + \nabla_a h_{bc} - \nabla_c h_{ab}\right).$$

We now compute the Riemann tensor, which is given by (2.2). Since the contraction of two Christoffel symbols is $\mathcal{O}(\epsilon^2)$ and the covariant derivative commutes on scalar fields, the Riemann tensor is

$$\begin{aligned} R_{abcd} &\approx \nabla_c \Gamma_{bda} - \nabla_d \Gamma_{bca} \\ &= \epsilon \frac{1}{2} \left(\nabla_c \nabla_d h_{ba} + \nabla_c \nabla_b h_{da} - \nabla_c \nabla_a h_{bd} - \nabla_d \nabla_c h_{ba} - \nabla_d \nabla_b h_{ca} + \nabla_d \nabla_a h_{bc} \right) \\ &= \epsilon \frac{1}{2} \left(\nabla_c \nabla_b h_{da} - \nabla_c \nabla_a h_{bd} - \nabla_d \nabla_b h_{ca} + \nabla_d \nabla_a h_{bc} \right), \end{aligned} \tag{2.6}$$

and then the Ricci tensor,

$$\begin{aligned} R_{bd} &= R^c{}_{bcd} \\ &\approx \epsilon \frac{1}{2} \left(\nabla_c \nabla_b h_d{}^c - \nabla_c \nabla^c h_{bd} - \nabla_d \nabla_b h_c{}^c + \nabla_d \nabla^c h_{bc} \right) \\ &= \epsilon \nabla^c \nabla_{(b} h_{d)c} - \epsilon \frac{1}{2} \left(\nabla^c \nabla_c h_{bd} + \nabla_d \nabla_b h \right). \end{aligned} \tag{2.7}$$

Consequently, the Ricci scalar is

$$\begin{aligned} R &= R^d{}_d \\ &\approx \epsilon \frac{1}{2} \left(\nabla^c \nabla^d h_{dc} - \nabla_c \nabla^c h - \nabla^d \nabla_d h + \nabla^d \nabla^c h_{dc} \right) \\ &= \epsilon \nabla^c \nabla^d h_{dc} - \epsilon \nabla^c \nabla_c h. \end{aligned}$$

Thus, the Einstein tensor becomes

$$\begin{aligned} G_{bd} &= R_{bd} - \frac{g_{bd}}{2} R \\ &\approx \epsilon \nabla^c \nabla_{(b} h_{d)c} - \epsilon \frac{1}{2} \left(\nabla^c \nabla_c h_{bd} + \nabla_d \nabla_b h \right) - \epsilon \frac{\eta_{bd}}{2} \left(\nabla^c \nabla^e h_{ec} - \nabla^c \nabla_c h \right). \end{aligned} \tag{2.8}$$

We can then compute the linearized Einstein tensor to be

$$\boldsymbol{H} = \frac{d}{d\epsilon} \boldsymbol{G} \bigg|_{\epsilon=0}.$$

Instead of writing these quantities in terms of the perturbation h_{cd}, we can rewrite them using trace-reversed metric $\bar{h}_{cd}$ via $h_{cd} = \bar{h}_{cd} + \eta_{cd} h/2$. With a similar computation to what was done before, the Ricci tensor then becomes

$$\begin{aligned}R_{bd} &\approx \epsilon\nabla^a\nabla_{(b}\bar{h}_{d)a} + \epsilon\frac{1}{2}\left(-2\nabla^a\nabla_{(b}\eta_{d)a}\frac{\bar{h}}{2} - \nabla^a\nabla_a\bar{h}_{bd} + \nabla^a\nabla_a\eta_{bd}\frac{\bar{h}}{2} + \nabla_b\nabla_d\bar{h}\right)\\ &= \epsilon\nabla^a\nabla_{(b}\bar{h}_{d)a} + \epsilon\frac{1}{2}\left(-\nabla^a\nabla_b\eta_{da}\frac{\bar{h}}{2} - \nabla^a\nabla_d\eta_{ba}\frac{\bar{h}}{2} - \nabla^a\nabla_a\bar{h}_{bd} + \nabla^a\nabla_a\eta_{bd}\frac{\bar{h}}{2} + \nabla_b\nabla_d\bar{h}\right)\\ &= \epsilon\nabla^a\nabla_{(b}\bar{h}_{d)a} + \epsilon\frac{1}{2}\left(-\nabla^a\nabla_a\bar{h}_{bd} + \eta_{bd}\nabla^a\nabla_a\frac{\bar{h}}{2}\right),\end{aligned}$$

and, similarly for the Ricci scalar,

$$R \approx \epsilon\nabla^a\nabla^d\bar{h}_{da} + \epsilon\frac{1}{2}\nabla^a\nabla_a\bar{h}.$$

The Einstein tensor finally is

$$\begin{aligned}G_{bd} &\approx \epsilon\nabla^a\nabla_{(b}\bar{h}_{d)a} + \epsilon\frac{1}{2}\left(-\nabla^a\nabla_a\bar{h}_{bd} + \eta_{bd}\nabla^a\nabla_a\frac{\bar{h}}{2}\right) - \epsilon\frac{\eta_{bd}}{2}\nabla^a\nabla^d\bar{h}_{da} - \epsilon\frac{\eta_{bd}}{4}\nabla^a\nabla_a\bar{h}\\ &= \epsilon\nabla^a\nabla_{(b}\bar{h}_{d)a} - \epsilon\frac{1}{2}\left(\nabla^a\nabla_a\bar{h}_{bd} + \eta_{bd}\nabla^a\nabla^e\bar{h}_{ea}\right),\end{aligned} \tag{2.9}$$

as desired. □

We can identify the operators in the previous theorem as follows:

$$\boldsymbol{H} = \mathcal{E}(\operatorname{div}\boldsymbol{h}) - \frac{1}{2}\left(\Box\boldsymbol{h} + \nabla^2 h\right) - \frac{\boldsymbol{\eta}}{2}\left(\operatorname{div}\operatorname{div}\boldsymbol{h} - \Box h\right),$$

or, in terms of the trace-reversed metric $\bar{\boldsymbol{h}}$,

$$\boldsymbol{H} = \mathcal{E}(\operatorname{div}\bar{\boldsymbol{h}}) - \frac{1}{2}\left(\Box\bar{\boldsymbol{h}} + \boldsymbol{\eta}\operatorname{div}\operatorname{div}\bar{\boldsymbol{h}}\right).$$

The differential operators appearing above are the following four-dimensional operators with signature $(-+++)$: $(\operatorname{div}\boldsymbol{h})_a = \nabla^b h_{ab}$, $\Box = \nabla^a\nabla_a$, $(\nabla^2 h)_{ab} = \nabla_a\nabla_b h$, and $(\mathcal{E}(\operatorname{div}\boldsymbol{h}))_{bd} = \nabla^c\nabla_{(b}h_{d)c}$.

From Theorem 2.3 with $\boldsymbol{T} = \epsilon\boldsymbol{\mathcal{T}} + \mathcal{O}(\epsilon^2)$, we see that the linearized Einstein equations are

$$\nabla^c\nabla_{(b}h_{d)c} - \frac{1}{2}\left(\nabla^c\nabla_c h_{bd} + \nabla_d\nabla_b h\right) - \frac{\eta_{bd}}{2}\left(\nabla^c\nabla^e h_{ec} - \nabla^c\nabla_c h\right) = 8\pi\mathcal{T}_{bd},$$

or in terms of the trace-reversed tensor

$$\nabla^a\nabla_{(b}\bar{h}_{d)a} - \frac{1}{2}\left(\nabla^a\nabla_a\bar{h}_{bd} + \eta_{bd}\nabla^a\nabla^e\bar{h}_{ea}\right) = 8\pi\mathcal{T}_{bd}, \tag{2.10}$$

with the energy-momentum tensor $T_{bd} = \epsilon\mathcal{T}_{bd} + \mathcal{O}(\epsilon^2)$. Finally, we see that $2\nabla_{(a}\xi_{b)}$, which appears in Theorem 2.2, is a solution to the homogeneous problem for any vector field $\boldsymbol{\xi}$. Indeed, we insert $2\nabla_{(a}\xi_{b)}$, expand the symmetric parts, interchange derivatives, and cancel corresponding terms,

$$\begin{aligned}
H_{bd} &= \frac{1}{2}\left(\nabla^c\nabla_b 2\nabla_{(d}\xi_{c)} + \nabla^c\nabla_d 2\nabla_{(b}\xi_{c)}\right) - \frac{1}{2}\left(\nabla^c\nabla_c 2\nabla_{(b}\xi_{d)} + \nabla_d\nabla_b 4\nabla^a\xi_a\right) \\
&\quad - \frac{\eta_{bd}}{2}\left(\nabla^c\nabla^a 2\nabla_{(a}\xi_{c)} - \nabla^c\nabla_c 4\nabla^a\xi_a\right) \\
&= \nabla^c\nabla_b\nabla_d\xi_c + \nabla^c\nabla_b\nabla_c\xi_d + \nabla^c\nabla_d\nabla_b\xi_c + \nabla^c\nabla_d\nabla_c\xi_b \\
&\quad - \nabla^c\nabla_c\nabla_b\xi_d - \nabla^c\nabla_c\nabla_d\xi_b - 2\nabla_d\nabla_b\nabla^a\xi_a \\
&= 0.
\end{aligned}$$

2.8 Gravitational Waves

In the special case of small perturbation, Einstein's equations can be written as a system of wave equations if we introduce the Lorentz gauge which we define below. This then enables us to study precisely gravitational waves and their polarization by further specializing to the Transverse-Traceless (TT) gauge, also introduced below.

Proposition 2.4. *For any trace-reversed metric, we can find a gauge-related trace-reversed metric $\bar{h}_{ab}$ satisfying the* Lorentz gauge*,*

$$\partial^a\bar{h}_{ab} = 0.$$

Moreover, the linearization of the Einstein tensor from Theorem 2.3 in terms of $\bar{h}_{ab}$ becomes

$$H_{bd} = -\Box\bar{h}_{bd},$$

where $\Box := \partial^a\partial_a$ is called the d'Alembertian*.*

Proof. Suppose we are given a metric h_{ab}. We use a mapping given by Theorem 2.2. By mapping $h_{ab}^{\text{new}} = h_{ab} - 2\partial_{(a}\xi_{b)}$, for any vector field ξ_b, from equation (2.5) we have that $\bar{h}_{ab}^{\text{new}} = \bar{h}_{ab} - (2\partial_{(a}\xi_{b)} - \eta_{ab}\partial^c\xi_c)$, and so $\partial^b\bar{h}_{ab}^{\text{new}} = \partial^b\bar{h}_{ab} - \partial^b\partial_b\xi_a$. We can take ξ_a satisfying the inhomogeneous wave equation, $\partial^b\partial_b\xi_a = \partial^b\bar{h}_{ab}$. Doing this, we realized that we found a diffeomorphism mapping the tensor h_{ab} to $\bar{h}_{ab}$ which is divergence-free and thus satisfies the Lorentz gauge condition. $\Box$

If the Lorentz gauge is satisfied, we then get that the metric satisfies

$$\Box \bar{h}_{bd} = -16\pi \mathcal{T}_{ab}, \tag{2.11}$$

from Theorem 2.4. We now seek plane wave solutions to the homogeneous equations.

Proposition 2.5. *Let k^a be a constant vector field, and A_{ab} a constant symmetric $(0,2)$-tensor field. We suppose that k^a is lightlike, namely that $k^c k_c = 0$, and that it belongs to the kernel of A_{ab}, so that $A_{ab}k^b = 0$. Let also $f(s)$ be a real-valued C^2 function of one-variable, and define a $(0,2)$-tensor field*

$$\bar{h}_{ab} := A_{ab} f(k_c x^c),$$

where x^c is the identity vector field on M_b and $k_c x^c$ is a scalar field. Then, $\bar{h}_{ab}$ is a plane wave solution in the Lorentz gauge to the homogeneous wave equation given by equation 2.11 with $\mathcal{T}_{ab} = 0$.

Proof. The plane wave satisfies

$$\Box A_{ab} f(k_c x^c) = A_{ab} \Box f(k_c x^c) = A_{ab} k_d k^d f''(k_c x^c) = 0,$$

since k^d is lightlike, and the Lorentz gauge condition,

$$\partial^a \bar{h}_{ab} = k^a A_{ab} f'(k_c x^c) = 0,$$

since k^a is in the kernel of A_{ab}. This concludes the proof. $\square$

We now introduce a specialization of the Lorentz gauge for a plane wave solution $\bar{h}_{ab} = A_{ab} f(k_c x^c)$ as given in Proposition 2.5: the *Transverse Traceless (TT) gauge*:

$$A^a{}_a = 0, \tag{2.12a}$$

$$A_{ab} u^b = 0, \tag{2.12b}$$

for some arbitrary constant timelike vector field u^a. For simplicity, we choose coordinates in which $u^a = (1,0,0,0)$ points along the time axis, and assume the wave propagates along the z-axis, so $k^a = k(1,0,0,1)$. In this gauge, we see that $\bar{h} = h = 0$, so $\bar{h}_{ab} = h_{ab}$.

Moreover, since $A_{ab}k^b = 0$, the tensor field A_{ab} must then take the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^+ & A^\times & 0 \\ 0 & A^\times & -A^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.13}$$

for arbitrary real numbers A^+ and A^-. These two numbers are the only two degrees of freedom left, and corresponds to two different polarizations of a gravitational wave, the $+$ and $\times$ polarization.

Proposition 2.6. *Consider the plane wave $\bar{h}_{ab} = A_{ab}f(k_c x^c)$ as given in Proposition 2.5 with $k^a = k(1,0,0,1)$. Then $\bar{h}_{ab}$ is gauge-related to a plane wave $\bar{h}'_{ab} = A'_{ab}f(k_c x^c)$ satisfying the TT gauge conditions* (2.12) *with $u^a = (1,0,0,0)$.*

Proof. Using equation (2.5), we can find a gauge-related metric via

$$\bar{h}'_{ab} := \bar{h}_{ab} - (2\partial_{(a}\xi_{b)} - \eta_{ab}\partial_c\xi^c), \tag{2.14}$$

for any vector field ξ^a. Now, we take a real-valued C^3 function $g(s)$ of one-variable with $g' = f$, and consider $\xi^a = B^a g(k_c x^c)$ with a constant vector field B^a. We see that $\bar{h}'_{ab}$ automatically satisfies the Lorentz gauge. Since k^a is lightlike and in the kernel of A_{ab}, from equation (2.14),

$$\begin{aligned} \partial^a\bar{h}'_{ab} = \partial^a\bar{h}_{ab} - \partial^a(2\partial_{(a}\xi_{b)} - \eta_{ab}\partial_c\xi^c) \\ = (k^a A_{ab} - k^a k_a B_b - k^a k_b B_a + k_b k_c B^c)f(k_d x^d) = 0, \end{aligned}$$

as desired.

Now, we take the special choice

$$\begin{aligned} B_0 &= \frac{1}{2k}(A_{00} + \frac{A}{2}), \\ B_j &= \frac{1}{k}(A_{j0} - k_j B_0). \end{aligned}$$

In order to satisfy the TT gauge, we need to verify the following two conditions.

First, to impose the traceless condition (2.12a), we require $0 = \bar{h}' = \bar{h} + 2\partial_a \xi^a$. Thus, ξ^a should satisfy $\partial_a \xi^a = -\frac{1}{2}\bar{h}$, or $k_a B^a = -\frac{1}{2}A$. This last condition holds, since

$$k_\mu B^\mu = -kB_0 - k\frac{1}{k}(A_{30} + kB_0) = -kB_0 - A_{30} - kB_0$$
$$= -2kB_0 + A_{00} = -A_{00} - \frac{A}{2} + A_{00} = -\frac{A}{2},$$

as desired.

Second, we impose the transverse condition (2.12b), $A_{\mu 0} = 0$. Indeed, from equation (2.14) with $\mu = 0$ and $\nu = 0$, we get

$$0 = A_{00} - 2kB_0 + k_\alpha B^\alpha = A_{00} - 2kB_0 - \frac{A}{2},$$

or $B_0 = \frac{1}{2k}(A_{00} + \frac{A}{2})$, as desired. We then turn to $\mu = j$ and $\nu = 0$, we get

$$0 = A_{j0} - kB_j - k_j B_0,$$

or $B_j = \frac{1}{k}(A_{j0} - k_j B_0)$ with B_0 found earlier, as desired.

We conclude that we have constructed a gauge-related plane wave solution $\bar{h}'_{ab}$ satisfying the TT gauge. $\square$

In order to find the force between two particles in free fall, we need to compute the acceleration between them. To do so, we study the geodesic deviation given by equation (2.3), as done in [1, p. 44]. Two particles initially at rest with separation vector field v^a have

$$a^\mu \approx -R^\mu{}_{0\nu 0} v^\nu.$$

We need to compute the Riemann tensor using equation (2.6). Because of the symmetries of the Riemann tensor and the metric, the only non-zero components in this case are

$$R_{\alpha 1 \beta 1} \approx -\frac{\epsilon}{2}\partial_\alpha \partial_\beta h_{11}, \tag{2.15a}$$
$$R_{\alpha 2 \beta 2} \approx -\frac{\epsilon}{2}\partial_\alpha \partial_\beta h_{22}, \tag{2.15b}$$
$$R_{\alpha 1 \beta 2} \approx -\frac{\epsilon}{2}\partial_\alpha \partial_\beta h_{12}, \tag{2.15c}$$

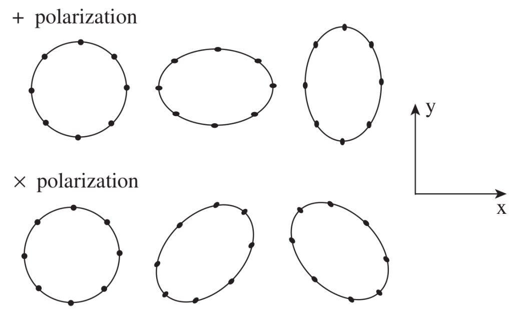


Figure 2.1: The effect of a gravitational wave with the given polarization passing through a ring of particles, [1, p. 45].

where α and β can only be 0 and 3. Thus, since $h_{\mu\nu} = \bar{h}_{\mu\nu}$ in this gauge,

$$a^1 \approx \frac{\epsilon}{2}\left(\ddot{h}_{11}v^1 + \ddot{h}_{12}v^2\right) = -\frac{\epsilon k^2}{2}\left(A^+v^1 + A^\times v^2\right) f(k_\alpha x^\alpha),$$
$$a^2 \approx \frac{\epsilon}{2}\left(\ddot{h}_{12}v^1 + \ddot{h}_{22}v^2\right) = -\frac{\epsilon k^2}{2}\left(A^\times v^1 - A^+v^2\right) f(k_\alpha x^\alpha).$$

We thus indeed observe the two independent polarizations, see Figure 2.1.

Chapter 3

EB System

In this chapter, we decompose the Riemann tensor R_{abcd} in a part M_{abcd} depending on the Ricci curvature and another tensor field W_{abcd} called the *Weyl tensor*,

$$R_{abcd} = M_{abcd} + W_{abcd}.$$

This decomposition is analogous to decomposing a matrix in terms of a traceless part and a multiple of the identity. Indeed, both M_{abcd} and W_{abcd} carry the symmetries of the Riemann tensor, but W_{abcd} is also fully traceless, while M_{abcd} depends only on the trace of the Riemann tensor, that is, on the Ricci tensor. In vacuum, general relativity tells us precisely that the Ricci curvature R_{ab} is zero. This implies that M_{abcd} is zero, and thus the Weyl tensor coincides with the Riemann tensor in vacuum. Our goal is then to find a method to recover the Weyl tensor. To do so, we follow [2, 3] and identify a first order hyperbolic system for the Weyl tensor. The first key ingredient is the second Bianchi identity satisfied by the Riemann tensor (2.4), and thus by the Weyl tensor in vacuum,

$$\nabla_{[a} W_{bc]de} = 0. \tag{3.1}$$

This identity allows us to identify a conservation law for the Weyl tensor. The second key ingredient is a decomposition of the Weyl tensor in terms of two parts: an electric and a magnetic part. Such decomposition was introduced in [11]. As it turns out, combining these two ingredients leads to a first order hyperbolic system for the Weyl tensor reminiscent to Maxwell's equations. A nonlinear evolution system based on the those two ingredients is discussed in [2] for vacuum. Moreover, a similar system using the second Bianchi identity, but based on four parts derived directly from the Riemann tensor, is presented in [3], and covers the non-vacuum case.

3.1 Volume Form and Hodge Duality

We call a smooth alternating $(0, p)$-tensor field a *differential p-form* or simply a *p-form*. The space of p-forms is denoted Λ^p. An n-dimensional manifold is said to be *orientable* if it has a nowhere vanishing n-form. If we define the following equivalence relation between nowhere vanishing n-forms on an orientable manifold: ω and η are equivalent if and only if $\omega = f\eta$ for some smooth function $f > 0$, then an *orientation* on M is a choice of equivalence class. A *volume form* is then any element of the equivalence class. A natural volume form is given by the *Levi-Civita tensor* whose expression in any coordinate system is given by

$$\omega_{i_1 \dots i_n} = \sqrt{|\det g|}\epsilon_{i_1 \dots i_n},$$

where $\det g$ is the determinant of the metric and $\epsilon_{i_1 \dots i_n}$ is the *Levi-Civita symbol*, with value $+1$ for an even permutation of the indices, -1 for an even permutation, and 0 otherwise. Unless otherwise noted, we refer to the natural volume form as the volume form.

A *positively oriented basis* of T_pM is a choice $v_1^a, \dots, v_n^a$ of ordered basis of T_pM. Two bases $(e_1, \dots, e_n)$ and $(f_1, \dots, f_n)$ are said to have the same orientation if the linear map $L : T_pM \to T_pM$ defined by $Le_i := f_i$ has positive determinant. The volume form $\omega_{a \dots b}(p)$ at $p \in M$ is the unique alternating $(0, n)$-tensor such that $\omega_{a \dots b}(p)v_1^a \dots v_n^b = 1$ for any positively oriented orthonormal (with respect to $g_{ab}(p)$) basis $v_1^a, \dots, v_n^a$ of T_pM. Assuming we have an orientable manifold M, the (natural) volume form $\omega_{a \dots b}$ for a manifold M with metric g_{ab} is the volume form $\omega_{a \dots b}(p)$ of T_pM relative to the inner product $g_{ab}(p)$ with the chosen orientation for any $p \in M$.

The volume form is related to *Hodge duality*, as detailed in [12]. Take nonnegative integers k and l such that $k + l = n$. Given a k-form $w_{a_1 \dots a_k}$, we define the *Hodge dual* of $w_{a_1 \dots a_k}$ to be the l-form

$$(\star \boldsymbol{w})_{b_1 \dots b_l} := \frac{1}{k!}\omega_{b_1 \dots b_l}{}^{a_1 \dots a_k} w_{a_1 \dots a_k}.$$

We also call the operator $\star$ from k-forms to l-forms the *Hodge star operator*. Using this definition and contractions of the volume form with itself given by

$$\omega^{a_1\ldots a_k c_1\ldots c_l}\omega_{c_1\ldots c_l b_1\ldots b_k} = (-1)^t\, k!l!\, \delta^{a_1\ldots a_k}_{b_1\ldots b_k},$$

where t is the number of negative eigenvalues of the metric, the Hodge dual of the Hodge dual is found to be

$$\begin{aligned}(\star\star\boldsymbol{w})_{a_1\ldots a_k} &= \frac{1}{k!l!}\omega_{a_1\ldots a_k}{}^{b_1\ldots b_l}\omega_{b_1\ldots b_l}{}^{c_1\ldots c_k}w_{c_1\ldots c_k}\\ &= \frac{(-1)^{kl}}{k!l!}\omega_{a_1\ldots a_k}{}^{b_1\ldots b_l}\omega^{c_1\ldots c_k}{}_{b_1\ldots b_l}w_{c_1\ldots c_k} = \frac{(-1)^{kl+t}}{k!l!}k!l!\,w_{a_1\ldots a_k} = (-1)^{kl+t}w_{a_1\ldots a_k}.\end{aligned}$$

The Hodge dual thus provides a bijective mapping between k- and l- forms. Now, we finally observe that the volume form is simply the Hodge dual of the constant function 1 viewed as a 0-tensor field, since

$$(\star 1)_{a_1\ldots a_n} = \omega_{a_1\ldots a_n}.$$

3.2 Weyl Tensor and Bel Decomposition

As introduced before, we decompose the Riemann tensor as $R_{abcd} = M_{abcd} + W_{abcd}$. The first tensor field is defined as

$$M_{abcd} := \frac{2}{n-2}\left[g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \frac{R}{n-1}g_{a[c}g_{d]b}\right],$$

following [1, p. 288] and [2]. Each of the two parts share the symmetries of the Riemann tensor given in Proposition 2.1: namely, they are skew-symmetric in each pair and satisfy the first Bianchi identity $R_{a[bcd]}$. The first tensor field depends on the Ricci curvature. The second tensor field is the *Weyl tensor* W_{abcd} and is fully traceless, $W^a{}_{bac} = 0$. It can then be shown that the Weyl tensor has $n(n+1)(n+2)(n-3)/12$ independent components when $n \geq 4$, and zero otherwise. In the particular case $n = 4$, the Weyl tensor thus has 10 independent components – and thus half of the components of the Riemann tensor.

We now specialize to the 4 dimensional case and introduce the Bel decomposition of the Weyl tensor. This decomposition is the second key ingredient needed in the search for a first order hyperbolic system for the Weyl tensor. We first note that, since the

Weyl tensor is skew-symmetric in each of the first and second pair of indices, we can view it as a *double 2-form*: an element of $\Lambda^2 \otimes \Lambda^2$. We can then apply the Hodge star operator to either of the two pairs (thus mapping a 2-form to a 2-form), and obtain the *dual Weyl tensors*,

$$W^*_{abcd} := (1 \otimes \star) W_{abcd} = \frac{1}{2} W_{abef} \omega^{ef}{}_{cd},$$
$$^*W_{abcd} := (\star \otimes 1) W_{abcd} = \frac{1}{2} \omega_{ab}{}^{ef} W_{efcd},$$

where ω_{abcd} is the volume form of the manifold. The dual of the dual Weyl tensor satisfies $W^{**}_{abcd} = -W_{abcd}$, and, using the interchange symmetry, $W^*_{abcd} = {}^*W_{cdab}$. If we fix any timelike unit vector n^a, we can divide the Weyl tensor into two parts: the *electric* and *magnetic* tensors

$$\bar{E}_{ab} := n^c n^d W_{cadb},$$
$$\bar{B}_{ab} := n^c n^d W^*_{cadb},$$

respectively. We call this decomposition of the Weyl tensor in terms of the electric and magnetic tensors the *Bel decomposition*, or the *electric-magnetic decomposition.* This decomposition is sometimes used to extract gravitational wave information out of the Weyl tensor after a numerical simulation has been run, [13] and [1, Sections 8.6-8.9]. It is also used to visualize the curvature tensor [14]. The electric and magnetic tensors inherit properties of the Weyl tensor.

Proposition 3.1. *The electric and magnetic tensors are symmetric, traceless, and orthogonal to* $\boldsymbol{n}$*.*

Proof. We start with the electric tensor $\bar{\boldsymbol{E}}$. From the interchange symmetry of the Weyl tensor, $\bar{\boldsymbol{E}}$ is symmetric. From the trace-free condition, $\bar{\boldsymbol{E}}$ is also trace-free. From the antisymmetry, $\boldsymbol{n} \cdot \bar{\boldsymbol{E}} = 0$, so $\bar{\boldsymbol{E}}$ is orthogonal to $\boldsymbol{n}$. This concludes the first part of the proof.

We turn to the magnetic tensor $\bar{\boldsymbol{B}}$. To show that the magnetic tensor $\bar{\boldsymbol{B}}$ is traceless, we compute

$$g^{ab} \bar{B}_{ab} = \frac{1}{2} n^c n^d W_{caef} g^{ab} \omega^{ef}{}_{db} = -\frac{1}{2} n^c n^d W_{caef} \omega^{aef}{}_{d} = -\frac{1}{2} n^c n^d W_{c[aef]} \omega^{aef}{}_{d} = 0,$$

by the first Bianchi identity. Now, we turn to the symmetry of $\bar{\boldsymbol{B}}$, by showing that the skew part is null. To do so, we define the *spatial volume form* $\omega_{abc} = \omega_{abcd}n^d$. We then compute

$$n_p\omega^{abpq}\bar{B}_{ab} = \frac{1}{2}n^c W_{ca}{}^{ef} n_p \omega^{abqp} n^d \omega_{efbd} = \frac{1}{2}n^c W_{ca}{}^{ef}\omega^{abq}\omega_{efb} = -\frac{1}{2}n^c W_{ca}{}^{ef}\omega^{baq}\omega_{bef}$$
$$= -\frac{1}{2}n^c W_{ca}{}^{ef}(\delta^a_e\delta^q_f - \delta^a_f\delta^q_e) = -\frac{1}{2}n^c W_{ca}{}^{aq} + \frac{1}{2}n^c W_{ca}{}^{qa} = 0,$$

by the trace-free condition, and $\bar{\boldsymbol{B}}$ is then symmetric. From the antisymmetry, $\boldsymbol{n}\cdot\bar{\boldsymbol{B}} = 0$, so $\bar{\boldsymbol{B}}$ is orthogonal to $\boldsymbol{n}$. This concludes the second part of the proof. □

The electric and magnetic tensors contribute all of the 10 independent components of the Weyl tensor. Indeed, we note the following two facts. First, a 2-tensor that is symmetric, traceless, and orthogonal to $\boldsymbol{n}$, has 5 independent components – and thus $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$ contribute up to 10 independent components. Second, with a straight forward calculation using the definitions of the electric and magnetic tensors, we see that the Weyl tensor can be written

$$W_{abcd} = 2\left[l_{a[c}\bar{E}_{d]b} - l_{b[c}\bar{E}_{d]a} - n_{[c}\bar{B}_{d]e}\omega^{e}{}_{ab} - n_{[a}\bar{B}_{b]e}\omega^{e}{}_{cd}\right], \tag{3.2}$$

with $l_{cd} = g_{cd} + 2n_c n_d$, and $\omega_{bcd} = \omega_{abcd}n^a$. Thus, $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$ are enough to recover the Weyl tensor.

3.3 Linearized Bel Decomposition

In the case of small perturbations, the EB system can be written as a constrained evolution system reminiscent of Maxwell's equations. This linearized decomposition and accompanying system were presented in [10], and we visit those results in the following. We now fix the background manifold $M_b = \mathbb{R}^4$ equipped the Minkowski metric $\boldsymbol{\eta}$. We suppose that we have a smooth one-parameter family $\boldsymbol{g}(\epsilon)$ of exact solutions to vacuum Einstein's equations $\boldsymbol{G}(\epsilon) = 0$ such that $\boldsymbol{g}(0) = \boldsymbol{\eta}$. We take a set of coordinates such that the metric $\boldsymbol{\eta}$ and its inverse have the form

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

In these coordinates, the volume form and the Levi-Civita symbol are equal, $\omega_{abcd} = \epsilon_{abcd}$. Moreover, we set $n^a = (1, 0, 0, 0)$. We write $W_{abcd} = C^0_{abcd} + \epsilon C_{abcd} + \mathcal{O}(\epsilon^2)$, where

$$C_{abcd} = \frac{d}{d\epsilon} W_{abcd}(\epsilon)\bigg|_{\epsilon=0},$$

and $C^0_{abcd} = 0$, since C^0_{abcd} corresponds to the linearized Weyl tensor of the background manifold M_b. We also set

$$\begin{aligned} E_{ab} &:= \frac{d}{d\epsilon}\bar{E}_{ab}(\epsilon)\bigg|_{\epsilon=0} = n^c n^d C_{cadb} = C_{0a0b}, \\ B_{ab} &:= \frac{d}{d\epsilon}\bar{B}_{ab}(\epsilon)\bigg|_{\epsilon=0} = n^c n^d C^*_{cadb} = C^*_{0a0b} = C_{0aef}\epsilon^{ef}{}_{0b}, \end{aligned}$$

and

$$\begin{aligned} H_{ab} &= {}^*C_{0a0b} = \frac{1}{2}\epsilon_{0a}{}^{ef}C_{ef0b}, \\ D_{ab} &= {}^*C^*_{0a0b} = \frac{1}{2}\epsilon_{0a}{}^{ef}C^*_{ef0b}. \end{aligned}$$

We call the four previous tensors the linearized *Bel decomposition* of $\boldsymbol{C}$. We now study some properties of $(0, 4)$-tensors that are skew-symmetric in the first and last pair.

Proposition 3.2. *Suppose we have a double 2-form $\boldsymbol{C}$ and set*

$$\begin{aligned} E_{ab} &:= C_{0a0b}, & B_{ab} &:= C^*_{0a0b} = C_{0aef}\epsilon^{ef}{}_{0b}, \\ H_{ab} &:= {}^*C_{0a0b} = \frac{1}{2}\epsilon_{0a}{}^{ef}C_{ef0b}, & D_{ab} &:= {}^*C^*_{0a0b} = \frac{1}{2}\epsilon_{0a}{}^{ef}C^*_{ef0b}. \end{aligned}$$

Then the matrix representation of $\boldsymbol{C}$ in the basis of bivectors $(\boldsymbol{t}\wedge\boldsymbol{x}, \boldsymbol{t}\wedge\boldsymbol{y}, \boldsymbol{t}\wedge\boldsymbol{z}, \boldsymbol{y}\wedge\boldsymbol{z}, \boldsymbol{z}\wedge\boldsymbol{x}, \boldsymbol{x}\wedge\boldsymbol{y})$, with coordinates taken from the background manifold M_b, is

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{H} & \boldsymbol{D} \end{pmatrix}.$$

Moreover, $\boldsymbol{C} \mapsto \boldsymbol{C}^*$ *is given by*

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{H} & \boldsymbol{D} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{B} & -\boldsymbol{E} \\ \boldsymbol{D} & -\boldsymbol{H} \end{pmatrix},$$

and $\boldsymbol{C} \mapsto \boldsymbol{C}^*$ *by*

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{H} & \boldsymbol{D} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{H} & \boldsymbol{D} \\ -\boldsymbol{E} & -\boldsymbol{B} \end{pmatrix},$$

and $\boldsymbol{C} \mapsto {}^*\boldsymbol{C}^*$ *by*

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{H} & \boldsymbol{D} \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{D} & -\boldsymbol{H} \\ -\boldsymbol{B} & \boldsymbol{E} \end{pmatrix}.$$

Proof. A double 2-form has $6 \cdot 6 = 36$ independent components, so that we can represent such tensor completely by the matrix

$$\begin{pmatrix} C_{0101} & C_{0102} & C_{0103} & C_{0123} & C_{0131} & C_{0112} \\ C_{0201} & C_{0202} & C_{0203} & C_{0223} & C_{0231} & C_{0212} \\ C_{0301} & C_{0302} & C_{0303} & C_{0323} & C_{0331} & C_{0312} \\ C_{2301} & C_{2302} & C_{2303} & C_{2323} & C_{2331} & C_{2312} \\ C_{3101} & C_{3102} & C_{3103} & C_{3123} & C_{3131} & C_{3112} \\ C_{1201} & C_{1202} & C_{1203} & C_{1223} & C_{1231} & C_{1212} \end{pmatrix},$$

as done in [10], further divide the representation into four blocks,

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{H} & \boldsymbol{D} \end{pmatrix}.$$

Since $C^*_{abcd} v^c w^d = C_{abef}(\epsilon^{ef}{}_{cd} v^c w^d)$ for any vector v^c and v^d, taking the dual of the Weyl tensor is equivalent to applying the Weyl tensor on the basis of bivectors permuted according to

$$\begin{aligned} \boldsymbol{t} \wedge \boldsymbol{x} &\mapsto \boldsymbol{y} \wedge \boldsymbol{z}, & \boldsymbol{y} \wedge \boldsymbol{z} &\mapsto -\boldsymbol{t} \wedge \boldsymbol{x}, \\ \boldsymbol{t} \wedge \boldsymbol{y} &\mapsto \boldsymbol{z} \wedge \boldsymbol{x}, & \boldsymbol{z} \wedge \boldsymbol{x} &\mapsto -\boldsymbol{t} \wedge \boldsymbol{y}, \\ \boldsymbol{t} \wedge \boldsymbol{z} &\mapsto \boldsymbol{x} \wedge \boldsymbol{y}, & \boldsymbol{x} \wedge \boldsymbol{y} &\mapsto -\boldsymbol{t} \wedge \boldsymbol{z}, \end{aligned}$$

which sends the first column of the partition to second, and the second to minus the first, so that the matrix representation of the dual tensor C^*_{abcd} is

$$\begin{pmatrix} \boldsymbol{B} & -\boldsymbol{E} \\ \boldsymbol{D} & -\boldsymbol{H} \end{pmatrix}.$$

Similarly, we note that ${}^*C_{bcde}$ sends the first row to second, and the second to minus the first,

$$\begin{pmatrix} \boldsymbol{H} & \boldsymbol{D} \\ -\boldsymbol{E} & -\boldsymbol{B} \end{pmatrix}.$$

Thus, the matrix representation of ${}^*C^*_{bcde}$ is

$$\begin{pmatrix} \boldsymbol{D} & -\boldsymbol{H} \\ -\boldsymbol{B} & \boldsymbol{E} \end{pmatrix},$$

as desired. □

Since the linearized Weyl tensor is skew-symmetric in each pair from Proposition 2.1, it is a double 2-form and the previous proposition applies directly.

Proposition 3.3. *Under the assumptions of Proposition 3.2, the following holds.*

- *The interchange symmetry is equivalent to:* $\boldsymbol{E}$ *and* $\boldsymbol{D}$ *are symmetric, and* $\boldsymbol{H} = \boldsymbol{B}^T$.
- *The null totally antisymmetric part condition is equivalent to:* $\operatorname{tr}\boldsymbol{B} + \operatorname{tr}\boldsymbol{H} = 0$.
- *The trace-free condition is equivalent to:* $\boldsymbol{E}$ *is trace-free,* $\boldsymbol{B}$ *and* $\boldsymbol{H}$ *are symmetric, and* $\boldsymbol{D} = -\boldsymbol{E}^T$.

Proof. The 16 conditions imposed by the first Bianchi identity breaks into $15 + 1$ conditions, the interchange symmetry and the null totally antisymmetric part condition, respectively. The interchange symmetry is equivalent to the symmetry of the matrix representation. The null totally antisymmetric part condition $C_{[abcd]} = 0$ has only 6 terms, since no index can be repeated. These 6 terms are the terms on the diagonals of $\boldsymbol{B}$ and $\boldsymbol{H}$, the only ones with no index repeated. Thus, the null totally antisymmetric part condition is equivalent to $\operatorname{tr}\boldsymbol{B} + \operatorname{tr}\boldsymbol{H} = 0$.

We now consider the 16 traceless conditions for C_{abcd}, divided into $1+3+3+6+3$. The singleton is $0 = C^a{}_{0a0} = E_{11} + E_{22} + E_{33}$. The first triplet is $0 = C^a{}_{0aj} =$

$C_{101j} + C_{202j} + C_{303j}$, which imposes the symmetry of $\boldsymbol{B}$. The second triplet is $0 = C^{a}{}_{ia0} = C_{1i10} + C_{2i20} + C_{3i30}$, which imposes the symmetry of $\boldsymbol{H}$. The 6-tuple is $0 = C^{a}{}_{iaj} = -C_{0i0j}$ for $i \neq j$, giving $E_{ij} = -D_{ji}$ for $i \neq j$. The last triplet is $0 = \sum_a C^{a}{}_{iai}$, from which we get $-E_{11} + D_{22} + D_{33} = 0$, $D_{11} - E_{22} + D_{33} = 0$, and $D_{11} + D_{22} - E_{33} = 0$, thus concluding that $\boldsymbol{D} = -\boldsymbol{E}^T + \boldsymbol{\delta} \operatorname{tr} \boldsymbol{E}/2 = -\boldsymbol{E}^T$. This concludes the proof. □

A similar result for the nonlinear case can be found in [3].

Proposition 3.2 thus leads to following matrix representation of the Weyl tensor,

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{B} \\ \boldsymbol{B} & -\boldsymbol{E} \end{pmatrix},$$

with $\boldsymbol{E}$ and $\boldsymbol{B}$ traceless and symmetric.

3.4 Linearized EB System

In this section, we find a constrained evolution system for $\boldsymbol{E}$ and $\boldsymbol{B}$ that is reminiscent of Maxwell's equations, as presented in [10], using the linearized version of the second Bianchi identity in vacuum given by (3.1),

$$\nabla_{[a} C_{bc]de} = 0.$$

This identity divides itself into 6 constraints that must be satisfied at all time, and 18 evolution equations. On one hand, if we pick the first three indices (a, b, c) to be $(1, 2, 3)$, then the corresponding 6 equations do not involve time derivatives and are therefore constraints. On the other hand, if we pick (a, b, c) as either $(0, 1, 2)$, $(0, 1, 3)$, $(0, 2, 3)$, then the corresponding $3 \times 6 = 18$ equations have time derivatives and are thus evolution equations. The following proposition details this decomposition.

Proposition 3.4. *The linearization of the second Bianchi identity* (3.1) *is equivalent to the constraints*

$$\operatorname{div} \boldsymbol{E} = 0,$$

$$\operatorname{div} \boldsymbol{B} = 0,$$

and the evolution equations,

$$\dot{\boldsymbol{E}} + \operatorname{curl} \boldsymbol{B} = 0, \tag{3.3}$$

$$\dot{\boldsymbol{B}} - \operatorname{curl} \boldsymbol{E} = 0. \tag{3.4}$$

We call this system, in which $\boldsymbol{E}$ and $\boldsymbol{B}$ are symmetric and trace-free matrix fields, the linearized Einstein-Bianchi system*, or simply the linearized* EB system*.*

Proof. We start with the vacuum second Bianchi identity, the first set of equations (3.1), written in the basis of the bivectors,

$$0 = \epsilon \nabla_{[\alpha} C_{\beta\gamma]\delta\theta} + \mathcal{O}(\epsilon^2) = \epsilon \partial_{[\alpha} C_{\beta\gamma]\delta\theta} + \mathcal{O}(\epsilon^2),$$

and so we get the vacuum second Bianchi identity for the linearized Weyl tensor,

$$\partial_{[\alpha} C_{\beta\gamma]\delta\theta} = 0. \tag{3.5}$$

We note that there are at most $4 \times 6 = 24$ independent conditions, due to the anti-symmetry in the triplet and the pair. We consider these 24 conditions in 4 groups. The first group is $(\alpha, \beta, \gamma) = (1, 2, 3)$ with (δ, θ) either $(0, 1)$, $(0, 2)$, or $(0, 3)$. The second group is $(\alpha, \beta, \gamma) = (1, 2, 3)$ with (δ, θ) either $(2, 3)$, $(3, 1)$, or $(1, 2)$. These first two groups have 3 equations each, do not have time derivatives, and are thus constraints. The third group is (α, β, γ) replaced by $(0, 2, 3)$, $(0, 3, 1)$, $(0, 1, 2)$, and (δ, θ) by $(0, 1)$, $(0, 2)$, and $(0, 3)$. The fourth group is (α, β, γ) replaced by $(0, 2, 3)$, $(0, 3, 1)$, $(0, 1, 2)$, and (δ, θ) by $(2, 3)$, $(3, 1)$, and $(1, 2)$. These last two groups have 9 equations each, have time derivatives, and are thus evolution equations.

We look into the first two groups. With (α, β, γ) replaced by $(1, 2, 3)$ in equation (3.5), we have

$$\partial_1 C_{23\delta\theta} + \partial_2 C_{31\delta\theta} + \partial_3 C_{12\delta\theta} = 0.$$

Then, further replacing (δ, θ) successively by $(0, 1)$, $(0, 2)$, and $(0, 3)$,

$$\partial_1 H_{1j} + \partial_2 H_{2j} + \partial_3 H_{3j} = 0,$$

so that

$$\operatorname{div} \boldsymbol{H}^T = 0.$$

Using the symmetries of the Weyl tensor given in Proposition 3.2, this is the first constraint, giving the first three equations,

$$\operatorname{div} \boldsymbol{B} = 0.$$

However, replacing (δ, θ), by $(2,3)$, $(3,1)$, and $(1,2)$, we get

$$\partial_1 D_{1j} + \partial_2 D_{2j} + \partial_3 D_{3j} = 0,$$

so that

$$\operatorname{div} \boldsymbol{D}^T = 0,$$

Using the symmetries of the Weyl tensor given in Proposition 3.2, this is the second constraint, giving the second three equations,

$$\operatorname{div} \boldsymbol{E} = 0.$$

We look into the next two groups. With (α, β, γ) in equation (3.5) replaced by $(0,2,3)$, $(0,3,1)$, $(0,1,2)$, and we get

$$\partial_0 C_{23\delta\theta} + \partial_2 C_{30\delta\theta} + \partial_3 C_{02\delta\theta} = 0.$$
$$\partial_0 C_{31\delta\theta} + \partial_3 C_{10\delta\theta} + \partial_1 C_{03\delta\theta} = 0,$$
$$\partial_0 C_{12\delta\theta} + \partial_1 C_{20\delta\theta} + \partial_2 C_{01\delta\theta} = 0.$$

On one hand, if we further take (δ, θ) to be $(0,1)$, $(0,2)$, and $(0,3)$, we get

$$\partial_0 H_{1j} - \partial_2 E_{3j} + \partial_3 E_{2j} = 0,$$
$$\partial_0 H_{2j} - \partial_3 E_{1j} + \partial_1 E_{3j} = 0,$$
$$\partial_0 H_{3j} - \partial_1 E_{2j} + \partial_2 E_{1j} = 0,$$

which we can combine into

$$\dot{\boldsymbol{H}}^T - \operatorname{curl} \boldsymbol{E}^T = 0.$$

Using the symmetries of the Weyl tensor given in Proposition 3.2, this is the evolution equations for $\boldsymbol{B}$, giving the second three equations,

$$\dot{\boldsymbol{B}} - \operatorname{curl} \boldsymbol{E} = 0.$$

On the other, if we take (δ, θ) instead to be $(2,3)$, $(3,1)$, and $(1,2)$, we get

$$\partial_0 D_{1j} - \partial_2 B_{3j} + \partial_3 B_{2j} = 0,$$
$$\partial_0 D_{2j} - \partial_3 B_{1j} + \partial_1 B_{3j} = 0,$$
$$\partial_0 D_{3j} - \partial_1 B_{2j} + \partial_2 B_{1j} = 0,$$

which can be combined into

$$\dot{\boldsymbol{D}}^T - \operatorname{curl} \boldsymbol{B}^T = 0,$$

Using the symmetries of the Weyl tensor given in Proposition 3.2, this is the evolution equations for $\boldsymbol{E}$, giving the second block of nine equations,

$$\dot{\boldsymbol{E}} + \operatorname{curl} \boldsymbol{B} = 0.$$

This concludes the proof. $\square$

Not only must $\boldsymbol{E}$ and $\boldsymbol{B}$ be divergence-free, as found from the previous proposition, but, from Proposition 3.2, they must also both be traceless and symmetric. We call *TSD* a matrix field that is traceless, symmetric, and divergence-free, at every point of the manifold.

Lemma 3.5. *Suppose we have a matrix field $\boldsymbol{A}$ in $H(\operatorname{curl}, \mathbb{M})$.*

- *The matrix field $\operatorname{curl} \boldsymbol{A}$ is divergence-free.*
- *If $\boldsymbol{A}$ is symmetric, then the matrix field $\operatorname{curl} \boldsymbol{A}$ is also trace-free.*
- *If, further, $\boldsymbol{A}$ is TSD, then the matrix field $\operatorname{curl} \boldsymbol{A}$ is TSD.*

Proof. The first point is clear, since $\operatorname{div} \operatorname{curl} \boldsymbol{A} = 0$.

Now, define

$$(\operatorname{vskw} \boldsymbol{A})_a := \frac{1}{2} \epsilon_a{}^{bc} A_{bc},$$

for any $(0,2)$-tensor field $\boldsymbol{A}$. This is the skew part of a matrix field seen as a vector. Then, the second point relies on the identity

$$\operatorname{tr} \operatorname{curl} \boldsymbol{A} = \epsilon^{abc} \partial_b A_{ac} = \partial_b \epsilon^{bca} A_{ac} = -2 \operatorname{div} \operatorname{vskw} \boldsymbol{A}$$

for any matrix field $\boldsymbol{A}$. In particular, if $\boldsymbol{A}$ is symmetric, then $\operatorname{vskw}\boldsymbol{A} = 0$ and the right hand side is zero. The matrix field $\boldsymbol{A}$ is then trace-free.

Finally, for any matrix field $\boldsymbol{A}$, we define

$$(\operatorname{curl}\boldsymbol{A})_{ab} := \epsilon_b{}^{cd}\partial_c A_{ad},$$

and compute

$$\begin{aligned}
&(2\operatorname{vskw}\operatorname{curl}\boldsymbol{A})_a \\
&\quad = \epsilon_a{}^{bc}\epsilon_c{}^{de}\partial_d A_{be} = \epsilon_{abc}\epsilon^{cde}\partial_d A^b{}_e = (\delta_a^d\delta_b^e - \delta_a^e\delta_b^d)\partial_d A^b{}_e = \partial_a A^b{}_b - \partial_b A^b{}_a \\
&\quad\quad = (\operatorname{grad}\operatorname{tr}\boldsymbol{A} - \operatorname{div}\boldsymbol{A}^T)_a = (\operatorname{grad}\operatorname{tr}\boldsymbol{A} - \operatorname{div}\boldsymbol{A} + 2\operatorname{div}\operatorname{skw}\boldsymbol{A})_a.
\end{aligned}$$

Thus, if $\boldsymbol{A}$ is TSD, the right hand side is zero, so that the matrix field $\operatorname{curl}\boldsymbol{A}$ is symmetric. This concludes the proof. □

Using this lemma, we now show that, if $\boldsymbol{E}$ and $\boldsymbol{B}$ are TSD at the initial time, they remain so for all time afterwards when propagated with this evolution equations in Proposition 3.4.

Proposition 3.6. *We are given sufficiently smooth TSD matrix fields $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$. If $\boldsymbol{E}(t)$ and $\boldsymbol{B}(t)$ satisfy $\boldsymbol{E}(0) = \boldsymbol{E}_0$ and $\boldsymbol{B}(0) = \boldsymbol{B}_0$ with the evolution equations given in Proposition 3.4 for all time $t \geq 0$, then $\boldsymbol{E}(t)$ and $\boldsymbol{B}(t)$ are TSD for $t \geq 0$.*

Proof. We need to show that $\operatorname{curl}\boldsymbol{E}$ is TSD, but this is clear by Lemma 3.5 since $\boldsymbol{E}$ is TSD. Similarly, $\operatorname{curl}\boldsymbol{E}$ is TSD. We can repeat the same proof with $\boldsymbol{E} \mapsto \boldsymbol{B}$ and $\boldsymbol{B} \mapsto -\boldsymbol{E}$. Therefore, the operator defined by

$$\begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}$$

maps TSD matrix fields to themselves, and so, by Theorem 4.5, $\boldsymbol{E}$ and $\boldsymbol{B}$ are TSD for all time. This concludes the proof. □

We thus have seen that the linearized second Bianchi identity in vacuum results in the constraints

$$\operatorname{div}\boldsymbol{E} = 0,$$

$$\operatorname{div}\boldsymbol{B} = 0,$$

and the evolution equations

$$\dot{\boldsymbol{E}} + \operatorname{curl} \boldsymbol{B} = 0,$$
$$\dot{\boldsymbol{B}} - \operatorname{curl} \boldsymbol{E} = 0,$$

in Proposition 3.4. Moreover, a solution to these evolution equations with initial conditions that are TSD must be TSD for all later times, as seen in Proposition 3.4. In Section 4.4, we will show that this system is well-posed.

In the nonlinear case discussed in [2], the second Bianchi identity can also be decomposed into the constraints

$$\operatorname{div} \bar{\boldsymbol{E}} - \bar{\boldsymbol{B}} \times\cdot\, \boldsymbol{K} = 0,$$
$$\operatorname{div} \bar{\boldsymbol{B}} - \bar{\boldsymbol{E}} \times\cdot\, \boldsymbol{K} = 0,$$

and evolution equations

$$\dot{\bar{\boldsymbol{E}}} + \operatorname{sym}\operatorname{curl} \bar{\boldsymbol{B}} + \boldsymbol{L}^1 = 0,$$
$$\dot{\bar{\boldsymbol{B}}} + \operatorname{sym}\operatorname{curl} \bar{\boldsymbol{E}} + \boldsymbol{L}^2 = 0,$$

with

$$\boldsymbol{L}^1 := -3\operatorname{sym}(\boldsymbol{K} \cdot \bar{\boldsymbol{E}}) + \bar{\boldsymbol{E}} \operatorname{tr} \boldsymbol{K} - \bar{\boldsymbol{E}} \times\times \boldsymbol{K} + 2\operatorname{sym}(\boldsymbol{a} \times \bar{\boldsymbol{B}}),$$
$$\boldsymbol{L}^2 := -3\operatorname{sym}(\boldsymbol{K} \cdot \bar{\boldsymbol{B}}) + \bar{\boldsymbol{B}} \operatorname{tr} \boldsymbol{K} - \bar{\boldsymbol{B}} \times\times \boldsymbol{K} + 2\operatorname{sym}(\boldsymbol{a} \times \bar{\boldsymbol{E}}),$$

where $\boldsymbol{K}$ is the extrinsic curvature of the manifold, $\boldsymbol{a} := \boldsymbol{n} \cdot \nabla \boldsymbol{n}$, and

$$(\boldsymbol{G} \times\cdot\, \boldsymbol{H})_j := \omega^{kl}{}_j G_{ik} H^i{}_l,$$
$$(\boldsymbol{G} \times\times \boldsymbol{H})_{ij} := \omega_i{}^{kl} G_{km} H_{ln} \omega_j{}^{mn},$$

for any $(0,2)$-tensor fields $\boldsymbol{G}$ and $\boldsymbol{H}$.

3.5 Gravitational Waves

We now find what plane waves in the TT gauge translate to in the EB system. Let k^a be a constant vector field, and A_{ab} a constant symmetric $(0,2)$-tensor field. We suppose that k^a is lightlike, namely that $k^c k_c = 0$, and that it belongs to the kernel of A_{ab}, so that $A_{ab}k^b = 0$. Let also $f(s)$ be a real-valued C^2 function of one-variable. We then have seen that we can find a plane wave solution $h_{ab} = A_{ab}f(k_c x^c)$ to the homogeneous wave equation given by equation 2.11 satisfying the TT gauge conditions (2.12). For simplicity, we assume the wave propagates along the z-axis, so $\boldsymbol{k} = k(1,0,0,1)$, and, from equation (2.13),

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^+ & A^\times & 0 \\ 0 & A^\times & -A^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(k_c x^c),$$

for arbitrary real numbers A^+ and A^-. In this case, since $W_{abcd} = R_{abcd}$ in vacuum, from equations (2.15), we have that the only non-zero components are

$$\begin{aligned} C_{\alpha 1\beta 1} &= -\frac{1}{2}\partial_\alpha\partial_\beta h_{11} = -\frac{1}{2}k_\alpha k_\beta A^+ f''(k_c x^c), \\ C_{\alpha 2\beta 2} &= -\frac{1}{2}\partial_\alpha\partial_\beta h_{22} = \frac{1}{2}k_\alpha k_\beta A^+ f''(k_c x^c), \\ C_{\alpha 1\beta 2} &= -\frac{1}{2}\partial_\alpha\partial_\beta h_{12} = -\frac{1}{2}k_\alpha k_\beta A^\times f''(k_c x^c), \end{aligned}$$

where α and β can only be 0 and 3. Thus, we have that a plane wave propagating along the z-axis gives rise to

$$\begin{aligned} E_{\mu\nu} = C_{0\mu 0\nu} &= \begin{pmatrix} C_{0101} & C_{0102} & C_{0103} \\ C_{0201} & C_{0202} & C_{0203} \\ C_{0301} & C_{0302} & C_{0303} \end{pmatrix} = \begin{pmatrix} -A^+ & -A^\times & 0 \\ -A^\times & A^+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{k^2}{2} f''(k_c x^c), \\ B_{\mu\nu} = C^*_{0\mu 0\nu} &= \begin{pmatrix} C_{0123} & C_{0131} & C_{0112} \\ C_{0223} & C_{0231} & C_{0212} \\ C_{0323} & C_{0331} & C_{0312} \end{pmatrix} = \begin{pmatrix} -A^\times & A^+ & 0 \\ A^+ & A^\times & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{k^2}{2} f''(k_c x^c), \end{aligned}$$

using the definition of $E_{\mu\nu}$ and $B_{\mu\nu}$. If $f \in C^3$, we then see that the electric and the magnetic tensors for a plane wave satisfy the linearized EB system given in Proposition 3.4. In particular, we see that these two tensor fields are TSD for all time.

Chapter 4

Mixed Abstract Hodge Wave Equations

We now consider a general framework for the discretization of first order linear hyperbolic systems. This framework enables the study, analysis, and discretization, of a large class of problems which includes acoustics, Maxwell's, and the linearized EB system. Because of the commonalities between these problems, the goal is to carry over the techniques to the linearized EB system.

In this chapter, we first review unbounded operators and Hilbert complexes. We then discuss well-posedness of first-order-in-time problems, along with a result on propagation of constraints. We then introduce the abstract Hodge wave equation. Then, following a review of Brezzi's theorem, the discretization of the abstract Hodge wave equation. We then turn to known polynomial complexes and to applications of the theory developed. Aiming at the linearized EB system, we first discuss application to the scalar wave equation, the vector wave equation, and Maxwell's equations. We in fact visit two different formulations of the linearized EB system based on two different complexes. The discretization of the first one is a direct application of the theory developed, but the discretization of the second involves the work of the next two chapters.

4.1 Unbounded Operators

Consider two Hilbert spaces X and Y with a linear operator $T : D(T) \to Y$ where $D(T) \subset X$. As an operator from X to Y, it is *not-necessarily-everywhere-defined* and *not-necessarily-bounded*. Instead of these two expressions, we often simply say that T is an *unbounded* operator. One should think of T as equipped with a given domain, so that changing the domain also changes the operator. Therefore, we say that two unbounded operators are *equal* if and only if (1) the domains are equal and (2) the actions agree. For instance, the gradient on $H^1(\mathbb{R})$ and the gradient on $\mathring{H}^1(\mathbb{R})$ are seen as two different operators. We say T is *densely defined* if $\overline{D(T)} = X$. Moreover, we say that the operator is *closed* if the graph $\Gamma(T) = \{(x, Tx) \mid x \in D(T)\}$ is closed in $X \times Y$. If T is closed, $D(T)$ is a Hilbert spaces with the graph norm, namely $\|v\|^2_{D(T)} = \|v\|^2 + \|dv\|^2$ for any $v \in D(T)$. By the Closed Graph theorem, if $D(T) = X$, then T is closed if and only if T is bounded. If T is a closed operator, we can then view T as either an unbounded linear operator between Hilbert spaces, or bounded operator on its domain with the graph norm.

We now define the *adjoint* $T^* : Y \to X$ of a densely defined unbounded operator $T : X \to Y$ between Hilbert spaces X and Y. We set the domain $D(T^*)$ of T^* to be the set of $w \in Y$ such that

$$|\langle w, Tv\rangle_Y| \leq c\|v\|_X,$$

for some constant c. With this domain, we can set $T^* : Y \to X$ to be the unique linear map such that

$$\langle T^* w, v\rangle_X = \langle w, Tv\rangle_Y$$

for $v \in D(T)$ and $w \in D(T^*)$. The adjoint of a densely defined unbounded operator is always closed. An unbounded operator $T : X \to X$ with dense domain $D(T)$ is said to be *self-adjoint* (or *skew-adjoint*) if $T^* = T$ (or $T^* = -T$). We also say that an unbounded operator is *symmetric* (or *skew-symmetric*) if $D(T) \subset D(T^*)$ and the action of T^* restricted to $D(T)$ agrees with T (or $-T$). In particular, an operator T is self-adjoint if and only if it is symmetric and $D(T^*) \subset D(T)$.

If we introduce the *rotated graph* of the adjoint of a densely defined unbounded operator $T : X \to Y$,

$$\widetilde{\Gamma}(T^*) = \{(x, y) \mid (y, -x) \in \Gamma(T^*)\} = \{(T^* y, -y) \mid y \in D(T^*)\},$$

then we have the relations

$$\begin{aligned} \Gamma(T)^\perp &= \widetilde{\Gamma}(T^*), \\ \overline{\Gamma(T)} &= \widetilde{\Gamma}(T^*)^\perp, \end{aligned}$$

and, if T is closed, then $\Gamma(T) = \widetilde{\Gamma}(T^*)^\perp$. The adjoint of a closed densely defined operator is also densely defined. Indeed, if $y \in Y$ and $y \perp D(T^*)$, then $(0, y) \in \widetilde{\Gamma}(T^*)^\perp = \Gamma(T)$ and so $y = 0$. Finally, for a closed densely defined operator $T : X \to Y$, we have

$$\mathcal{R}(T)^\perp = \mathcal{N}(T^*), \tag{4.1a}$$

$$\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}, \tag{4.1b}$$

$$\mathcal{R}(T^*)^\perp = \mathcal{N}(T), \tag{4.1c}$$

$$\mathcal{N}(T^*)^\perp = \overline{\mathcal{R}(T)}. \tag{4.1d}$$

4.2 Hilbert Complexes

In this section, we review Hilbert complexes and follow [15, 5]. A *Hilbert complex* (W, d) is a sequence of Hilbert spaces W^k

$$0 \xrightarrow{d} W^0 \xrightarrow{d} W^1 \xrightarrow{d} \dots \xrightarrow{d} W^n \xrightarrow{d} 0,$$

with an associated sequence of closed densely defined linear operators d^k from W^k to W^{k+1} such that $d^{k+1} \circ d^k v = 0$ for any $v \in V^k$, where we set V^k as the domain of d^k for any k. We then define the associated bounded complex (V, d),

$$0 \to V^0 \xrightarrow{d} V^1 \xrightarrow{d} \dots \xrightarrow{d} V^n \to 0,$$

called the *domain complex*. This complex is said to be bounded, since the operators d are bounded on their corresponding domain V with the graph norm. As an example, we can consider the *smooth differential k-forms* $\Lambda^k(\Omega) := C^\infty(\Omega, \mathbb{R}^{n\times\cdots\times n}_{\text{skw}})$ for any integer

$k \geq 0$. We can equip them with the *exterior derivatives* $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$. The domains of the exterior derivatives are

$$H\Lambda^k := \left\{\omega \in L^2\Lambda^k \mid d\omega \in L^2\Lambda^{k+1}\right\}$$

for $k \geq 0$. The associated Hilbert spaces are $L^2\Lambda^k := L^2(\Omega, \mathbb{R}^{n\times\cdots\times n}_{\mathrm{skw}})$. On a three dimensional domain Ω, this translates into the *de Rham complex*,

$$0 \to H^1(\mathbb{R}) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}, \mathbb{V}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}, \mathbb{V}) \xrightarrow{\mathrm{div}} L^2(\mathbb{R}) \to 0. \tag{4.2}$$

If we set $L^2(\mathbb{X})$ to be the set of square-integrable functions from Ω to $\mathbb{X}$, then the Hilbert spaces associated to the previous sequences are $L^2(\mathbb{R})$, $L^2(\mathbb{V})$, $L^2(\mathbb{V})$, and $L^2(\mathbb{R})$. The complex is a closed Hilbert complex. The sequence is exact on a contractible domain Ω, except in the first position.

We call elements of the range $\mathfrak{B}^k = dV^{k-1}$ *k-coboundaries* (or simply *boundaries* when the context is clear), and elements of the null space $\mathfrak{Z}^k = \mathcal{N}(d^k)$, *$k$-cocycles* (or simply *cycles*). We note that $\mathfrak{B}^k \subset \mathfrak{Z}^k$, since $d \circ d = 0$, so we can form $\mathcal{H}^k := \mathfrak{Z}^k/\mathfrak{B}^k$, the *$k$th cohomology space*. We say that the Hilbert complex is *closed*, whenever the operators have closed range, so that $\mathfrak{B}^k$ is closed in W^k for all k (or, equivalently, in V^k since the norms for W^k and V^k are equivalent for $\mathfrak{Z}^k$ which contains $\mathfrak{B}^k$).

We denote by d_k^* from $W^k \to W^{k-1}$ the adjoint of d^{k-1}. We denote the domain of d_k^* by V_k^*, so that we can define the *dual complex* (W, d^*),

$$0 \leftarrow V_0^* \xleftarrow{d^*} V_1^* \xleftarrow{d^*} \ldots \xleftarrow{d^*} V_n^* \leftarrow 0$$

which is a bounded chain complex. If (W, d) is a closed Hilbert complex, then the dual complex (W, d^*) also is. The dual complex of the de Rham complex is

$$0 \leftarrow L^2(\mathbb{R}) \xleftarrow{-\mathrm{div}} \mathring{H}(\mathrm{div}, \mathbb{V}) \xleftarrow{\mathrm{curl}} \mathring{H}(\mathrm{curl}, \mathbb{V}) \xleftarrow{-\mathrm{grad}} \mathring{H}^1(\mathbb{R}) \leftarrow 0. \tag{4.3}$$

This is also a closed Hilbert complex. The associated Hilbert spaces are still $L^2(\mathbb{R})$, $L^2(\mathbb{V})$, $L^2(\mathbb{V})$, and $L^2(\mathbb{R})$. We remind that, since the domains are different, the operators appearing here are different than the ones in the de Rham complex.

Assuming that $\mathfrak{B}^k$ is closed, we now identify the cohomology space $\mathcal{H}^k$ with the subspace $\mathfrak{H}^k \subset \mathfrak{Z}^k$, the orthogonal complement of $\mathfrak{B}^k$ inside $\mathfrak{Z}^k$. Thus, by definition, we have the orthogonal decomposition $\mathfrak{Z}^k = \mathfrak{B}^k \oplus \mathfrak{H}^k$. Moreover, using equations (4.1), since $\mathfrak{B}^*_k$ are closed, we have $\mathfrak{B}^*_k = (\mathfrak{Z}^k)^\perp$, so that

$$\mathfrak{H}^k := (\mathfrak{B}^k)^{\perp_{\mathfrak{Z}^k}} = \mathfrak{Z}^k \cap (\mathfrak{B}^k)^\perp = \mathfrak{Z}^k \cap \mathfrak{Z}^*_k = \left\{ u \in V^k \cap V^*_k \mid du = 0,\, d^*u = 0 \right\}.$$

We call the elements of $\mathfrak{H}^k$ *harmonic k-forms*. The name comes from the fact that harmonic k-forms form precisely the null space of the *Hodge Laplacian* $L^k = d^{k-1}d^*_k + d^*_{k+1}d^k$ with domain

$$D(L^k) = \left\{ u \in V^k \cap V^*_k \mid du \in V^*_{k+1}, d^*u \in V^{k-1} \right\}.$$

We say that a Hilbert complex has the *compactness property* if $V^k \cap V^*_k$ is a dense and compact subspace of W^k for all k. A Hilbert complex with the compactness property is also *Fredholm*: the harmonic k-forms are finite dimensional for all k. A Fredholm Hilbert complex is always closed.

The following orthogonal decompositions are a matter of using the previous observations on $W^k = \mathfrak{Z}^k \oplus (\mathfrak{Z}^k)^\perp$ and $V^k = \mathfrak{Z}^k \oplus (\mathfrak{Z}^k)^{\perp_{V^k}}$.

Theorem 4.1 (Hodge decomposition)**.** *Any closed Hilbert complex satisfies the orthogonal decompositions*

$$\begin{aligned} W^k &= \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}^*_k, \\ V^k &= \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus (\mathfrak{Z}^k)^{\perp_{V^k}}. \end{aligned}$$

We further say that a complex is exact if $\mathfrak{B}^k = \mathfrak{Z}^k$ for all k. In that case, $\mathcal{H}^k = 0$ and $\mathfrak{H}^k = 0$ for all k. In the case of the de Rham complex on a contractible domain, the complex is exact and

$$\begin{aligned} L^2(\mathbb{V}) &= \operatorname{grad} H^1(\mathbb{R}) \oplus \operatorname{curl} \mathring{H}(\operatorname{curl}, \mathbb{V}), \\ L^2(\mathbb{V}) &= \operatorname{curl} H(\operatorname{curl}, \mathbb{V}) \oplus \operatorname{grad} \mathring{H}^1(\mathbb{R}), \\ L^2(\mathbb{R}) &= \operatorname{div} H(\operatorname{div}, \mathbb{V}). \end{aligned}$$

Now, since d^k is an isomorphism from $(\mathfrak{Z}^k)^{\perp_V}$ onto $\mathfrak{B}^{k+1}$, d^k has a bounded inverse, we have the following result.

Theorem 4.2 (Poincaré inequality)**.** *For each k, there exists a constant C such that*

$$\|z\|_V \le C\|dz\|$$

for any $z \in (\mathfrak{Z}^k)^{\perp_V}$.

4.3 Well-Posedness and Constraint Propagation

To show well-posedness of an abstract wave equation we introduce in the next section, we need to introduce a corollary of the Hille-Yosida Theorem [16, Proposition 4.1.6 and Corollary 2.4.9].

Theorem 4.3. *Given a skew-adjoint operator $\mathcal{L}$ with domain $D(\mathcal{L})$ dense in X, $U_0 \in D(\mathcal{L})$, and F in both $C([0,T],X)$ and at least one of $W^{1,1}([0,T],X)$ and $L^1((0,T),D(\mathcal{L}))$. There exists a unique solution*

$$U \in C^0([0,T],D(\mathcal{L})) \cap C^1([0,T],X)$$

to the system $\dot{U} = \mathcal{L}U + F$ with $U(0) = U_0$.

Moreover, we are also interested in the case in which we need to restrict $\mathcal{L}$ to a closed subspace of X.

Theorem 4.4. *Suppose we have a Hilbert space X and a closed subspace K, and a skew-adjoint operator $\mathcal{L}$ with domain $D(\mathcal{L})$ dense in X, such that*

- *$P_K D(\mathcal{L}) \subset D(\mathcal{L})$, with P_K the orthogonal projection onto K,*
- *$D(\mathcal{L}) \cap K$ is dense in K,*
- *$\mathcal{L}(D(\mathcal{L}) \cap K) \subset K$.*

If we define the operator $\widetilde{\mathcal{L}} : K \to K$ with domain $D(\widetilde{\mathcal{L}}) := D(\mathcal{L}) \cap K$ by restricting $\mathcal{L}$ to K, then $\widetilde{\mathcal{L}}$ is skew-adjoint.

Proof. First, let us show that $\mathcal{L}(D(\mathcal{L}) \cap K^\perp) \subset K^\perp$. Take $u \in D(\mathcal{L}) \cap K^\perp$ and $v \in D(\mathcal{L}) \cap K$, then

$$\langle \mathcal{L}u, v\rangle = -\langle u, \mathcal{L}v\rangle = 0,$$

using that $\mathcal{L}(D(\mathcal{L}) \cap K) \subset K$. Thus, $\mathcal{L}(D(\mathcal{L}) \cap K^\perp) \subset (D(\mathcal{L}) \cap K)^\perp$. Since $D(\mathcal{L}) \cap K$ is dense in K, $\mathcal{L}(D(\mathcal{L}) \cap K^\perp) \subset K^\perp$.

We claim that $\widetilde{\mathcal{L}}$ is skew-adjoint. For any u and v in $D(\widetilde{\mathcal{L}})$,

$$\langle u, \widetilde{\mathcal{L}}v\rangle_K = \langle u, \mathcal{L}v\rangle_X = -\langle \mathcal{L}u, v\rangle_X = -\langle \widetilde{\mathcal{L}}u, v\rangle_K,$$

by the definition of $\mathcal{L}$, since $D(\widetilde{\mathcal{L}}) \subset D(\mathcal{L})$. This shows that $D(\widetilde{\mathcal{L}}) \subset D(\widetilde{\mathcal{L}}^*)$ and $\mathcal{L}^* = -\mathcal{L}$ on $D(\widetilde{\mathcal{L}})$. Thus, $\widetilde{\mathcal{L}}$ is automatically skew-symmetric. We now show that $D(\widetilde{\mathcal{L}}^*) \subset D(\widetilde{\mathcal{L}})$. Take $u \in D(\widetilde{\mathcal{L}}^*) \subset K$, for any $w \in D(\mathcal{L})$,

$$\langle u, \mathcal{L}w \rangle = \langle u, \widetilde{\mathcal{L}} P_K w \rangle + \langle u, \mathcal{L} P_{K^\perp} w \rangle = \langle u, \widetilde{\mathcal{L}} P_K w \rangle \leq C \|P_K w\| \leq C\|w\|,$$

using $\mathcal{L}(D(\mathcal{L}) \cap K^\perp) \subset K^\perp$, $P_K D(\mathcal{L}) \subset D(\mathcal{L})$, and

$$|\langle u, \widetilde{\mathcal{L}} v \rangle|_X \leq C\|v\|_X,$$

for any $v \in D(\widetilde{\mathcal{L}})$. Therefore, $u \in D(\mathcal{L}^*)$, and so $D(\widetilde{\mathcal{L}}^*) \subset D(\mathcal{L}^*) \cap K = D(\widetilde{\mathcal{L}})$. This then shows that $\widetilde{\mathcal{L}}$ is skew-adjoint, as desired. $\square$

If we combine the previous two results, we understand the well-posedness of an operator restricted to a subspace.

Corollary 4.5. *Suppose the hypotheses of Theorem 4.4 hold. If we also have that $U_0 \in D(\widetilde{\mathcal{L}})$, and F in both $C([0,T],K)$ and at least one of $W^{1,1}([0,T],K)$ and $L^1((0,T),D(\widetilde{\mathcal{L}}))$, then there exists a unique solution*

$$U \in C^0([0,T], D(\widetilde{\mathcal{L}})) \cap C^1([0,T],K)$$

to the system $\dot{U} = \mathcal{L}U + F$ with $U(0) = U_0$.

Proof. From Theorem 4.4, we know that $\widetilde{\mathcal{L}}$ is skew-adjoint. We can now apply Theorem 4.3 on $\widetilde{\mathcal{L}}$. Therefore, there exists a unique solution

$$U \in C^0([0,T], D(\widetilde{\mathcal{L}}) \cap C^1([0,T],K)$$

to the system

$$\dot{U} = \widetilde{\mathcal{L}}U + F$$

with $U(0) = U_0$. Since $D(\widetilde{\mathcal{L}}) \subset D(\mathcal{L})$ and $K \subset X$, U is also the unique solution to the system

$$\dot{U} = \mathcal{L}U + F$$

with $U(0) = U_0$, which exists by Theorem 4.3 on $\mathcal{L}$. This concludes the proof. $\square$

We can apply this corollary to show the propagation of constraints. We thus want the solution to remain in the kernel $K := \ker \mathcal{M}$ of some not-necessarily-bounded operator $\mathcal{M} : X \to Y$, for Hilbert spaces X and Y. Since the kernel of a (not-necessarily-bounded) linear operator is closed, the conclusion of the theorem says that $\mathcal{M}U = 0$ on $[0, T]$.

4.4 Abstract Hodge Wave

We are now ready to present the abstract framework for an abstract wave equation associated to a complex. We have Hilbert spaces W^0, W^1, W^2, equipped with closed unbounded closed range operator d^i from W^i to W^{i+1} and dense domains $D(d^i) = V^i$. The *first order abstract Hodge wave equation in mixed form* is the system

$$\dot{\sigma} = d^* u, \tag{4.4a}$$

$$\dot{u} = -d\sigma - d^* \rho, \tag{4.4b}$$

$$\dot{\rho} = du. \tag{4.4c}$$

We introduce the operator $\mathcal{L}$ from $\boldsymbol{W} := W^0 \times W^1 \times W^2$ to itself, with domain $D(\mathcal{L}) = V^0 \times (V^1 \cap V_1^*) \times V_2^*$, given by

$$\mathcal{L}(\sigma, u, \rho) = (-d^* u, d\sigma + d^* \rho, -du). \tag{4.5}$$

We then set $\xi = (\sigma, u, \rho)$ and write the abstract Hodge wave equation as

$$\dot{\xi} + \mathcal{L}\xi = 0.$$

If the initial data $\xi_0 = (\sigma_0, u_0, \rho_0)$ is in $D(\mathcal{L})$, then we say that the (strong) mixed abstract Hodge wave equation (4.4) has a *strong solution* ξ if and only if $\xi \in C^0([0, T], D(\mathcal{L})) \cap C^1([0, T], \boldsymbol{W})$, ξ satisfies the equations (4.4), and $\xi(0) = \xi_0$. The following theorem says that there exists a unique strong solution.

Theorem 4.6. *Let W^0, W^1, W^2 be Hilbert spaces. For $i = 0, 1$, let $d^i : W^i \to W^{i+1}$ be a closed unbounded operator with dense domains V^i and closed range with the complex property, $d \circ d = 0$. Define $\mathcal{L}$ as in (4.5) with $D(\mathcal{L}) := V^0 \times (V^1 \cap V_1^*) \times V_2^*$. Given initial data (σ_0, u_0, ρ_0) in $D(\mathcal{L})$, there exists a unique solution*

$$(\sigma, u, \rho) \in C^0([0, T], D(\mathcal{L})) \cap C^1([0, T], \boldsymbol{W})$$

to the mixed abstract Hodge wave equation (4.4) *satisfying the initial conditions* $\sigma(0) = \sigma_0$, $u(0) = u_0$*, and* $\rho(0) = \rho_0$.

Proof. If we show that $\mathcal{L}$ is skew-adjoint, by Theorem 4.3, there exists a unique solution in $C^0([0,T], D(\mathcal{L})) \cap C^1([0,T], \boldsymbol{W})$ to the mixed abstract Hodge wave equation (4.4) for the given initial conditions in $D(\mathcal{L})$, as desired. We show that $\mathcal{L}$ is skew-adjoint following an argument from Arnold.

We show that $D(\mathcal{L}) = D(\mathcal{L}^*)$ and $\mathcal{L}^*(\sigma, u, \rho) = -\mathcal{L}(\sigma, u, \rho)$ for any $(\sigma, u, \rho) \in D(\mathcal{L})$. Thus, we first note that, for any (σ, u, ρ) and (τ, v, μ) in $D(\mathcal{L})$,

$$\begin{aligned}
\langle (\sigma, u, \rho), \mathcal{L}(\tau, v, \mu) \rangle &= -\langle \sigma, d^* v \rangle + \langle u, d\tau + d^*\mu \rangle - \langle \rho, dv \rangle \\
&= \langle d^* u, \tau \rangle - \langle d\sigma + d^*\rho, v \rangle + \langle du, \mu \rangle \\
&= -\langle \mathcal{L}(\sigma, u, \rho), (\tau, v, \mu) \rangle,
\end{aligned}$$

which shows that $D(\mathcal{L}) \subset D(\mathcal{L}^*)$ and $\mathcal{L}^* = -\mathcal{L}$ on $D(\mathcal{L})$.

It remains to show that $D(\mathcal{L}^*) \subset D(\mathcal{L})$. We consider $(\sigma, u, \rho) \in D(\mathcal{L}^*)$, i.e. there exists a constant C such that

$$|\langle (\sigma, u, \rho), \mathcal{L}(\tau, v, \mu) \rangle| \leq C \|\tau, v, \mu\|_{\boldsymbol{W}}, \tag{4.6}$$

for any $(\tau, v, \mu) \in D(\mathcal{L})$.

We show that $u \in V \cap V^*$. Taking $v = 0$ and $\mu = 0$ in equation (4.6) gives that $|\langle u, d\tau \rangle| \leq C\|\tau\|$ for any $\tau \in V$. Thus, $u \in V^*$. Similarly, taking $v = 0$ and $\tau = 0$ instead shows that $u \in V$, since d is a closed operator and so $d^{**} = d$. Therefore, $u \in V \cap V^*$, as desired.

We now show that $\sigma \in V$. Consider any $w \in V^*$. We know that $w \in W$ has an orthogonal decomposition $w = z + y$, where $z \in \mathfrak{Z}$, the kernel of d, and $y \in \mathfrak{Z}^\perp = \mathfrak{B}^*$, the range of d^*. Now, $z = w - y \in V^*$, but $z \in V$, so $z \in V \cap V^*$. Thus, by equation (4.6) with $(\tau, v, \mu) = (0, z, 0) \in D(\mathcal{L})$, recalling that $dz = 0$, $|\langle \sigma, d^* z \rangle| \leq C\|z\|$. However, using the complex property $d \circ d = 0$, we have $d^* w = d^* z$ and $\|z\| \leq \|w\|$, so $|\langle \sigma, d^* w \rangle| \leq C\|w\|$. Thus, $\sigma \in V$.

The argument is similar for $\rho \in V^*$, so $(\sigma, u, \rho) \in D(\mathcal{L})$, and so $D(\mathcal{L}^*) \subset D(\mathcal{L})$. This concludes the proof. $\square$

We now write the mixed abstract Hodge wave equation in weak form, and to do so we introduce $\boldsymbol{V} = V^0 \times V^1 \times W^2$. We thus seek $(\sigma, u, \rho) \in C^0([0,T], \boldsymbol{V}) \cap C^1([0,T], \boldsymbol{W})$ such that

$$(\dot{\sigma}, \tau) = (u, d\tau), \tag{4.7a}$$

$$(\dot{u}, v) = -(d\sigma, v) - (\rho, dv), \tag{4.7b}$$

$$(\dot{\rho}, \mu) = (du, \mu), \tag{4.7c}$$

for any $(\tau, v, \mu) \in \boldsymbol{V}$. The initial conditions are taken in $\boldsymbol{V}$. We can define the bilinear form $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ as

$$a(\sigma, u, \rho; \tau, v, \mu) = -(u, d\tau) + (d\sigma, v) + (\rho, dv) - (du, \mu),$$

for any $(\sigma, u, \rho) \in \boldsymbol{V}$, and $(\tau, v, \mu) \in \boldsymbol{V}$. Thus, we write the weak mixed abstract Hodge wave equation (4.7) as finding $\xi \in C^0([0,T], \boldsymbol{V}) \cap C^1([0,T], \boldsymbol{W})$ such that

$$(\dot{\xi}, \psi) + a(\xi, \psi) = 0, \tag{4.8}$$

for any $\psi \in \boldsymbol{V}$. We say that the mixed Hodge wave equation (4.4) have a *weak solution* $\xi = (\sigma, u, \rho)$ if $\xi = (\sigma, u, \rho) \in C^0([0,T], \boldsymbol{V}) \cap C^1([0,T], \boldsymbol{W})$, $\xi = (\sigma, u, \rho)$ satisfies the equations (4.7), and $\xi(0) = (\sigma, u, \rho)(0) \in \boldsymbol{V}$. To see the uniqueness of a weak solution, we suppose we have two solutions $\xi_1 = (\sigma_1, u_1, \rho_1)$ and $\xi_2 = (\sigma_2, u_2, \rho_2)$. The difference satisfies the Hodge wave equation with zero initial conditions. By taking $\tau = \sigma_1 - \sigma_2$, $v = u_1 - u_2$, and $\mu = \rho_1 - \rho_2$, we get that $\|\sigma_1 - \sigma_2\|^2 + \|u_1 - u_2\|^2 + \|\rho_1 - \rho_2\|^2 = 0$. Thus, the two solutions are the same. The following theorem shows that the strong and weak forms are equivalent in the given sense.

Theorem 4.7. *Under the hypothesis of Theorem 4.6, the strong and weak forms* (4.4) *and* (4.7) *of the mixed abstract Hodge wave are equivalent in the following sense.*

- *If $\xi = (\sigma, u, \rho) \in C^0([0,T], D(\mathcal{L})) \cap C^1([0,T], \boldsymbol{W})$ is a solution to the strong form* (4.4) *with initial conditions in $D(\mathcal{L})$, then $\xi = (\sigma, u, \rho) \in C^0([0,T], \boldsymbol{V}) \cap C^1([0,T], \boldsymbol{W})$ is also a solution to the weak form* (4.7) *with the same initial conditions.*
- *If $\xi = (\sigma, u, \rho) \in C^0([0,T], \boldsymbol{V}) \cap C^1([0,T], \boldsymbol{W})$ is a solution to the weak form* (4.7) *with initial conditions in $D(\mathcal{L})$, then $\xi = (\sigma, u, \rho) \in C^0([0,T], D(\mathcal{L})) \cap C^1([0,T], \boldsymbol{W})$ and is also a solution to the strong form* (4.4) *with the same initial conditions.*

Proof. On one hand, if ξ is a strong solution to the mixed abstract Hodge wave equation (4.4), it certainly belongs to $C^0([0,T],\boldsymbol{V}) \cap C^1([0,T],\boldsymbol{W})$. The weak form (4.7) then easily follows from the strong form (4.4), and so the strong solution is also a weak solution. Thus, the first statement of this theorem certainly holds.

On the other hand, suppose that ξ is a weak solution to the mixed abstract Hodge wave equation (4.4) with $\xi(0) \in D(\mathcal{L})$. By Theorem 4.7, there exists a strong solution $\widetilde{\xi}$ to the mixed abstract Hodge wave equation (4.4) taking on the same initial data $\xi(0) \in D(\mathcal{L})$. By the first part of this theorem, that solution also satisfies the weak form. However, we have seen that uniqueness holds for the initial value problem for the weak form. Thus, ξ must coincide with $\widetilde{\xi}$, so that $\xi \in C^0([0,T],D(\mathcal{L})) \cap C^1([0,T],\boldsymbol{W})$ and is a strong solution. Therefore, the second statement of this theorem also holds. This concludes the proof. $\square$

4.5 Brezzi's Theorem

Before proceeding with the discretization of the Hodge wave equation, let us first review an essential result in the theory of mixed finite element. We consider a Hilbert space X. We denote the dual space of X by X^*. We say that a bilinear functional is *continuous* (or *bounded*) if there exists $C < \infty$ such that $|a(u,v)| \leq C\|u\|\|v\|$ for all $(u,v) \in X \times X$. We also say that a bilinear functional is *coercive* if there exists $\alpha > 0$ such that $a(u,u) \geq \alpha\|u\|^2$ for all $u \in X$.

Theorem 4.8 (Lax-Milgram)**.** *Let $a : X \times X \to \mathbb{R}$ be a continuous and coercive bilinear functional, and $f \in X^*$. Then there exists a unique $u \in X$ such that $a(u,v) = \langle f, v\rangle$ for all $v \in X$.*

We follow [17]. We say that a bilinear form a satisfies the *inf-sup condition* if

$$\inf_{0\neq u\in X} \sup_{0\neq v\in X} \frac{a(u,v)}{\|u\|\|v\|} > 0.$$

Equivalently, a satisfies this condition if there exists $\gamma > 0$ such that for all $0 \neq u \in X$ there exists $0 \neq v \in X$ such that

$$a(u,v) \geq \gamma\|u\|\|v\|.$$

We say that the bilinear form a satisfies the *dense range condition* if for all $0 \neq v \in X$ there exists $u \in X$ such that $a(u,v) \neq 0$. Equivalently, a satisfies this condition if for any $u \in X$, if $a(u,v) = 0$ for any $v \in X$ then $u = 0$.

Theorem 4.9. *Suppose we have a continuous bilinear form a on $X \times X$. Then a satisfies the inf-sup and dense range conditions if and only if for any $f \in X^*$ there exists $u \in X$ such that $a(u,v) = \langle f, v \rangle$ for any $v \in X$.*

In this case, if we define $L : X \to X^*$ such that $Lu = a(u, \cdot)$, we see that

$$\inf_{0 \neq u \in X} \sup_{0 \neq v \in X} \frac{a(u,v)}{\|u\|\|v\|} = \|L^{-1}\|^{-1}_{\mathcal{L}(X^*,X)}.$$

Suppose we have a bounded bilinear form a on $X \times X$, with which we define a linear functional $Lu = a(u, \cdot)$, which satisfies the inf-sup condition. Suppose further that we have a finite dimensional subspace X_h of X on which we restrict the bilinear form a to define $L_h : X_h \to X_h^*$. We want to find an approximation u_h by solving $L_h u_h = F_h$ to the solution u of $Lu = F$, where F_h is the restriction to X_h of the bounded linear functional F on X.

Theorem 4.10 (Quasi-optimality of the Finite Element Methods)**.** *Suppose the bilinear form a is bounded on $X_h \times X_h$ and satisfies the inf-sup condition over $X_h \times X_h$ with a uniform lower bound $\gamma > 0$. Then there exists a constant C independent of h, u and u_h such that*

$$\|u - u_h\| \leq C \inf_{v \in X_h} \|u - v\|.$$

Proof. Let $r_h u$ be the orthogonal projection of u in X_h, so that

$$\|u - r_h u\| = \inf_{v \in X_h} \|u - v\|.$$

Since the bilinear form a satisfies the inf-sup condition over $X_h \times X_h$, we can use Theorem 4.9, and the uniform bound on the inf-sup constant $\gamma \leq \|L_h^{-1}\|^{-1}_{\mathcal{L}(X_h^*,X_h)}$. Therefore,

$$\begin{aligned}
\|u - u_h\| &\leq \|u - r_h u\| + \|r_h u - u_h\| \\
&\leq \|u - r_h u\| + \gamma^{-1} \|L_h \left(r_h u - u_h\right)\|_{X_h^*} \\
&= \|u - r_h u\| + \gamma^{-1} \|L \left(r_h u - u\right)\|_{X_h^*},
\end{aligned}$$

since $L_h(r_h u - u_h) = L(r_h u - u_h) = L(r_h u - u) + L(u - u_h) = L(r_h u - u)$ on X_h, Now, L is bounded on X with constant M, so

$$\begin{aligned}\|u - u_h\| &\leq \left(1 + M\gamma^{-1}\right) \|r_h u - u\| \\ &\leq \left(1 + M\gamma^{-1}\right) \inf_{v \in X_h} \|u - v\|.\end{aligned}$$

This concludes the proof. □

We now consider a second Hilbert space Y, and a second bounded linear map $B : Y \to X^*$, along with bounded linear functionals F and G on X and Y, respectively. We then consider the abstract saddle point problem: find $u \in X$, $p \in Y$ such that

$$\begin{aligned} Lu + B^* p &= F, \\ Bu &= G. \end{aligned}$$

Defining $b(u, q) = Bu(q)$, we can also write the problem as: find $u \in X$, $p \in Y$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= F(v), \\ b(u, q) &= G(q), \end{aligned}$$

for any $v \in X$, $q \in Y$. We say that this problem is *well-posed* if there exists a unique solution $(u, p) \in X \times Y$ and some constant C such that

$$\|u\|_X + \|p\|_Y \leq C \left(\|F\|_{X^*} + \|G\|_{Y^*}\right).$$

We then present a corresponding well-posedness theorem. We set $Z \subset X$ to be the kernel of B, namely $Z = \{u \in X \mid b(u, q) = 0 \, \forall q \in Y\}$.

Theorem 4.11 (Brezzi [18])**.** *Suppose*

- *a satisfies both the inf-sup and dense range conditions over $Z \times Z$,*
- *b satisfies the inf-sup condition on $X \times Y$.*

Then the saddle point problem is well-posed.

4.6 Discretization

We now define the discretized version of equations (4.7). We take $\boldsymbol{V}_h = V_h^0 \times V_h^1 \times V_h^2$ to be a finite dimensional subspace of $\boldsymbol{V}$. We make the following assumptions on the spaces V_h^i, for $0 \le i \le 2$. First, we assume *density*: for any i, for any $v \in V^i$, $\lim_{h\to 0} \inf_{w \in V_h^i} \|v - w\|_{V^i} = 0$. This hypothesis is essential if we hope to have convergence of a finite element method. Second, we need the discrete spaces V_h^0, V_h^1, and V_h^2 to satisfy the *subcomplex property*: the spaces form a chain complex (V_h, d), with the operators $d^i|_{V_h^i}$ which we denote simply d. In other words, $dV_h^0 \subset V_h^1$ and $dV_h^1 \subset W_h^2$. This hypothesis enables us to mimic the analysis performed at the continuous level at the discrete level. Third, we assume the existence of *bounded cochain projections*: for each i, we have a bounded linear map $\pi_h^i : V^i \to V_h^i$ restricting to the identity on V_h^i such that

$$\begin{tikzcd} V^0 \arrow[r, "d"] \arrow[d, "\pi_h^0"] & V^1 \arrow[r, "d"] \arrow[d, "\pi_h^1"] & W^2 \arrow[d, "\pi_h^2"] \\ V_h^0 \arrow[r, "d"] & V_h^1 \arrow[r, "d"] & V_h^2 \end{tikzcd}$$

commutes. These hypotheses are standard in the framework of Finite Element Exterior Calculus and are important in carrying the analysis from the continuous to the discrete level.

Moreover, the following simple observation is needed in what follows.

Lemma 4.12. *Given nonnegative functions $F \in C^0([0,T])$ and $Q \in C^1([0,T])$, such that*

$$\frac{d}{dt} Q^2 \le FQ,$$

we have

$$Q(t) \le Q(0) + \frac{1}{2}\int_0^t F(s)ds,$$

for $t \in [0,T]$.

The estimate is sharp in the sense that taking $Q(t) = t$ and $F(t) = 2$ for all t satisfies the hypothesis, and, in that case, the asserted inequality is an equality.

The method is then as follows. We seek $(\sigma_h, u_h, \rho_h) \in C^1(\boldsymbol{V}_h)$ such that

$$(\dot\sigma_h, \tau) = (u_h, d\tau), \tag{4.9a}$$

$$(\dot u_h, v) = -(d\sigma_h, v) - (\rho_h, dv), \tag{4.9b}$$

$$(\dot\rho_h, \mu) = (du_h, \mu), \tag{4.9c}$$

for any $(\tau, v, \mu) \in \boldsymbol{V}_h$, with initial conditions $\boldsymbol{V}_h$. We can write the discrete weak mixed abstract Hodge wave equation (4.9) as finding $\xi_h \in C^1(\boldsymbol{V}_h)$ such that

$$(\dot{\xi}_h, \psi) + a(\xi_h, \psi) = 0 \tag{4.10}$$

for any $\psi \in \boldsymbol{V}_h$. We say that $\xi_h = (\sigma_h, u_h, \rho_h)$ is a *discrete solution* to the mixed abstract Hodge wave equation (4.4) if $\xi_h \in C^1([0,T], \boldsymbol{V}_h)$, ξ_h satisfies the equations (4.9), and $\xi_h(0) \in \boldsymbol{V}_h$. We know the equations (4.9) have a unique solution, since they form a square system of linear ordinary differential equations.

We call $\xi := (\sigma, u, \rho)$ the weak solution to the mixed abstract Hodge wave equation (4.7), and $\xi_h := (\sigma_h, u_h, \rho_h)$ the discrete solution. We define $d(\sigma, u, \rho) = (0, d\sigma, du)$ for any $(\sigma, u, \rho) \in \boldsymbol{V}$.

In order to get convergence of the method, we need the following elliptic projection $\Pi_h \xi(t) = \xi_h^{\Pi}(t) \in \boldsymbol{V}_h$ of $\xi(t) \in \boldsymbol{V}$ such that

$$(\xi_h^{\Pi}, \psi) + a(\xi_h^{\Pi}, \psi) = (\xi, \psi) + a(\xi, \psi), \tag{4.11}$$

for any $\psi \in \boldsymbol{V}_h$. We denote $\Pi_h \xi(t) = \Pi_h(\sigma, u, \rho)(t) = (\sigma_h^{\Pi}, u_h^{\Pi}, \rho_h^{\Pi})(t)$. The system for the elliptic projection (4.11) is non-singular. Indeed, we see that the system is square, so we only need to show that $(\sigma, u, \rho) = 0$ implies that $(\sigma_h^{\Pi}, u_h^{\Pi}, \rho_h^{\Pi}) = 0$. This is clear, since, taking $\psi = \xi_h^{\Pi}$ in (4.11), we get $\|\sigma_h^{\Pi}\|^2 + \|u_h^{\Pi}\|^2 + \|\rho_h^{\Pi}\|^2 = 0$. Let us know define the bilinear form $A : \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$ by

$$A(\sigma, u, \rho; \tau, v, \mu) = (\sigma, \tau) + (u, v) + (\rho, \mu) + a(\sigma, u, \rho; \tau, v, \mu).$$

which is uniformly bounded in h on $\boldsymbol{V}_h \times \boldsymbol{V}_h$. We want to show the stability of this bilinear form.

Proposition 4.13. *Suppose we have $\boldsymbol{V} = V^0 \times V^1 \times V^2$ with the subcomplex property. Then, the bilinear form A satisfies the inf-sup condition over $\boldsymbol{V}$ with lower bound $\gamma = 1/\sqrt{12}$.*

Proof. We consider any $(\sigma, u, \rho) \in \boldsymbol{V}$. We then take $(\tau, v, \mu) = (\sigma, u + d\sigma, \rho - du) \in \boldsymbol{V}$, so that

$$\begin{aligned} A(\sigma,u,\rho;\tau,v,\mu) &= A(\sigma,u,\rho;\sigma,u,\rho) + A(\sigma,u,\rho;0,d\sigma,-du) \\ &= \|\sigma\|^2 + \|u\|^2 + \|\rho\|^2 + (u,d\sigma) + \|d\sigma\|^2 - (\rho,du) + \|du\|^2 \\ &\geq \|\sigma\|^2 + \|u\|^2 + \|\rho\|^2 - \frac{1}{2}\|u\|^2 - \frac{1}{2}\|d\sigma\|^2 + \|d\sigma\|^2 \\ &\quad - \frac{1}{2}\|\rho\|^2 - \frac{1}{2}\|du\|^2 + \|du\|^2 \\ &\geq \frac{1}{2}(\|\sigma\|_V^2 + \|u\|_V^2 + \|\rho\|_W^2) = \frac{1}{2}\|\sigma,u,\rho\|_{\boldsymbol{V}}^2. \end{aligned}$$

Since $\|\tau,v,\mu\|_{\boldsymbol{V}} \leq \sqrt{3}\|\sigma,u,\rho\|_{\boldsymbol{V}}$, we see that

$$A(\sigma,u,\rho;\tau,v,\mu) \geq \frac{1}{\sqrt{12}}\|\sigma,u,\rho\|_{\boldsymbol{V}}\|\tau,v,\mu\|_{\boldsymbol{V}},$$

and the inf-sup condition is satisfied. □

Moreover, this result also holds at the discrete level by the same proof and with the same constant, as long as our subspaces form a subcomplex. In particular, the inf-sup constant is uniformly bounded in h. By the quasi-optimality result given in Theorem 4.10, we have that there exists a constant C such that

$$\|\zeta - \Pi_h\zeta\|_{\boldsymbol{V}} \leq C \inf_{\psi\in\boldsymbol{V}_h} \|\zeta-\psi\|_{\boldsymbol{V}},$$

for any $\zeta \in \boldsymbol{V}$.

The system for the bilinear form can be recognize as a discretization of the Hodge Laplacian with a lower order term,

$$(dd^* + d^*d + I)u = f.$$

The convergence of the discretization of the Hodge Laplacian for the case without the lower order term is well-known [4, 5], and requires a Hilbert complex with the compactness property. However, it is an upcoming result from Arnold and Li that the standard estimates still hold with this particular lower order term.

We now reduce the question of convergence of the method to the convergence of the elliptic projection. For convenience, we introduce the projected error $\varepsilon_h := \xi_h^\Pi - \xi_h = (\varepsilon_h^\sigma, \varepsilon_h^u, \varepsilon_h^\rho)$, with $\varepsilon_h^\sigma = \sigma_h^\Pi - \sigma_h$, $\varepsilon_h^u = u_h^\Pi - u_h$, and $\varepsilon_h^\rho = \rho_h^\Pi - \rho_h$.

Theorem 4.14. *Suppose $\boldsymbol{V}_h = V_h^0 \times V_h^1 \times V_h^2$ is a finite dimensional subspace of $\boldsymbol{V}$ with the subcomplex property. We have*

$$\|\xi - \xi_h\|_{L^\infty(\boldsymbol{W})} \leq \|\xi_h^\Pi(0) - \xi_h(0)\| + \|\xi_h^\Pi - \xi\|_{L^\infty(\boldsymbol{W})} \\ + (1+T)(\|(\xi_h^\Pi - \xi)(0)\|_{\boldsymbol{W}} + \|\dot{\xi}_h^\Pi - \dot{\xi}\|_{L^1(\boldsymbol{W})}).$$

Proof. From 4.8 and 4.10, we get

$$(\dot{\xi}, \psi) + a(\xi, \psi) = (\dot{\xi}_h, \psi) + a(\xi_h, \psi),$$

for any $\psi \in \boldsymbol{V}_h$, and also

$$(\dot{\xi}_h^\Pi - \dot{\xi}_h, \psi) + a(\xi_h^\Pi - \xi_h, \psi) = (\dot{\xi}_h^\Pi - \dot{\xi}, \psi) + a(\xi_h^\Pi - \xi, \psi),$$

or, in terms of the projected error ε_h,

$$(\dot{\varepsilon}_h, \psi) + a(\varepsilon_h, \psi) = (\dot{\xi}_h^\Pi - \dot{\xi}, \psi) + a(\xi_h^\Pi - \xi, \psi) = (\dot{\xi}_h^\Pi - \dot{\xi}, \psi) - (\xi_h^\Pi - \xi, \psi).$$

where we used the definition of the elliptic projection (4.11) at the final step. Since $a(\psi, \psi) = 0$ for any $\psi \in \boldsymbol{V}_h$, taking $\psi = \varepsilon_h \in \boldsymbol{V}_h$, we see that

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_h\|^2 = (\dot{\xi}_h^\Pi - \dot{\xi}, \varepsilon_h) - (\xi_h^\Pi - \xi, \varepsilon_h) \leq (\|\dot{\xi}_h^\Pi - \dot{\xi}\| + \|\xi_h^\Pi - \xi\|)\|\varepsilon_h\|.$$

By Lemma 4.12,

$$\|\varepsilon_h(t)\| \leq \|\varepsilon_h(0)\| + \int_0^t (\|\dot{\xi}_h^\Pi - \dot{\xi}\| + \|\xi_h^\Pi - \xi\|),$$

or

$$\|\varepsilon_h(t)\|_{\boldsymbol{W}} \leq \|\varepsilon_h(0)\| + (\|\dot{\xi}_h^\Pi - \dot{\xi}\|_{L^1(\boldsymbol{W})} + \|\xi_h^\Pi - \xi\|_{L^1(\boldsymbol{W})}).$$

Since $|f(t)| \leq |f(0)| + |\int_0^t \dot{f}|$,

$$\|\varepsilon_h(t)\|_{\boldsymbol{W}} \leq \|\varepsilon_h(0)\| + (1+T)(\|(\xi_h^\Pi - \xi)(0)\|_{\boldsymbol{W}} + \|\dot{\xi}_h^\Pi - \dot{\xi}\|_{L^1(\boldsymbol{W})}).$$

Taking the supremum over time,

$$\|\varepsilon_h\|_{L^\infty(\boldsymbol{W})} \leq \|\varepsilon_h(0)\| + (1+T)(\|(\xi_h^\Pi - \xi)(0)\|_{\boldsymbol{W}} + \|\dot{\xi}_h^\Pi - \dot{\xi}\|_{L^1(\boldsymbol{W})}),$$

and using triangle inequality conclude the proof. □

Now, if we have a Hilbert complex with the compactness property, the elliptic projection converges. Moreover, if we take initial data $\xi_h(0) = \xi_h^\Pi(0)$, then the method for the Hodge wave equation converges.

4.7 Polynomial de Rham Complex

We now want to consider examples building up to the linearized EB system: the acoustic wave equation, the vector wave equation and Maxwell's equations, and then the matrix wave equation and the linearized EB system. To do this, we consider the special case of the theory developed for the Hodge wave equation applied to the n-dimensional de Rham complex (4.2). For a discretization $\mathcal{T}_h$, we recall two families Λ_h of finite element spaces presented in the Finite Element Exterior Calculus framework: $\mathcal{P}\Lambda$ and $\mathcal{P}^-\Lambda$, as presented in [5]. We discuss their shape functions and then their degrees of freedom [19].

The shape functions for the assembled space $\mathcal{P}_r\Lambda^k$ on any simplex T are given by

$$\mathcal{P}_r\Lambda^k(T) = \left\{ \sum_{\sigma\in\Sigma(k,n)} p_\sigma dx^\sigma \mid p_\sigma \in \mathcal{P}_r(T) \right\},$$

the space of differential k-forms on each simplex T with polynomial coefficients of degree at most r. We define $\mathcal{P}_r\Lambda^k(T)$ to be zero if $r < 0$, $k < 0$, or $k > n$. A key property is that this family form a subcomplex of the de Rham complex,

$$0 \to \mathcal{P}_r\Lambda^0(T) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(T) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(T) \to 0,$$

since $d\mathcal{P}_r\Lambda^k(T) \subset \mathcal{P}_{r-1}\Lambda^{k+1}(T)$. We can similarly define $\mathcal{H}_r\Lambda^k(T)$, the space of differential k-forms with homogeneous polynomial coefficients of degree r.

We introduce the *Koszul differential operator* κ mapping k-forms to $(k-1)$-forms by

$$(\kappa\boldsymbol{\omega})_{a\ldots b}(x) = \omega_{ca\ldots b}(x)x^c,$$

where x^c is a vector of $\mathbb{R}^n$ that was identified with x. The Koszul operator can be thought of as contraction with x. With this in hand, the shape functions for the assembled space $\mathcal{P}_r^-\Lambda^k(T)$ on any simplex T are given by the space

$$\mathcal{P}_r^-\Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}(T).$$

In particular, $\mathcal{P}_r^-\Lambda^0(T) = \mathcal{P}_r\Lambda^0(T)$ and $\mathcal{P}_r^-\Lambda^n(T) = \mathcal{P}_{r-1}\Lambda^n(T)$. Moreover, we define $\mathcal{P}_r^-\Lambda^k(T)$ to be zero if $r \leq 0$, $k < 0$, $k > n$. The operators d and κ are related together by the *Homotopy formula*,

$$(\kappa d + d\kappa)\omega = (k+r)\omega$$

for $\omega \in \mathcal{H}_r\Lambda^k(T)$. From this formula follows

$$\mathcal{H}_r\Lambda^k = \kappa\mathcal{H}_{r-1}\Lambda^{k+1}(T) \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}(T).$$

Therefore, we have

$$d\mathcal{P}_r^-\Lambda^k(T) = d\mathcal{P}_r\Lambda^k(T) \subset \mathcal{P}_{r-1}\Lambda^{k+1}(T) \subset \mathcal{P}_r^-\Lambda^{k+1}(T),$$

and so the family $\mathcal{P}^-\Lambda(T)$ also forms a complex

$$0 \to \mathcal{P}_r^-\Lambda^0(T) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(T) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(T) \to 0.$$

Moreover, given a degree r high enough, the two families can be combined together to build 2^{n-1} discretizations of the de Rham complex. Indeed, we start with $\mathcal{P}_r\Lambda^0(T) = \mathcal{P}_r^-\Lambda^0(T)$, and then we have have two choices at every level until the last level where we can only choose $\mathcal{P}_{r-n}\Lambda^n(T) = \mathcal{P}_{r-n+1}^-\Lambda^n(T)$. These sequences are called *polynomial de Rham complexes.*

With the two spaces of shape functions in hand, we can now present their degrees of freedom. A unisolvent set of degrees of freedom for $\mathcal{P}_r\Lambda^k(T)$ is

$$\omega \mapsto \int_f (\operatorname{tr}_f \omega) \wedge \mu$$

for $\omega \in \mathcal{P}_r\Lambda^k(T)$, $\mu \in \mathcal{P}_{r+k-d}^-\Lambda^{d-k}(f)$, $f \in \Delta_d(T)$, and $d \geq k$. A unisolvent set of degrees of freedom for $\mathcal{P}_r^-\Lambda^k(T)$ are

$$\omega \mapsto \int_f (\operatorname{tr}_f \omega) \wedge \mu$$

for $\omega \in \mathcal{P}_r^-\Lambda^k(T)$, $\mu \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f)$, $f \in \Delta_d(T)$, and $d \geq k$. Given a triangulation $\mathcal{T}_h$, these degrees of freedom give the assembled spaces the exact continuity to be in $H\Lambda^k$, since

$$\mathcal{P}_r\Lambda^k := \mathcal{P}_r\Lambda^k(\mathcal{T}_h) := \left\{\omega \in H\Lambda^k \mid \omega|_T \in \mathcal{P}_r\Lambda^k(T)\right\},$$

and similarly for $\mathcal{P}_r^-\Lambda^k$. These degrees of freedom also define canonical cochain projections $C\Lambda^k(\bar{\Omega}) \to \mathcal{P}_r\Lambda^k$. Combining these projections with smoothers as in [5], it is possible to construct bounded cochain projections $\pi_h^k : L^2\Lambda^k(\Omega) \to \Lambda_h^k$ with the following estimates.

Theorem 4.15 ([5]). *Given a domain Ω with triangulation $\mathcal{T}_h$, the projections $\pi_h^k : L^2\Lambda^k \to \Lambda_h^k$ satisfy the following.*

- *Let Λ_h^k be one of the spaces $\mathcal{P}_{r+1}^-\Lambda^k$ for $r \geq 0$ or $\mathcal{P}_r\Lambda^k$ for $r \geq 1$. Then, π_h^k is a projection onto Λ_h^k and satisfies*

$$\|\omega - \pi_h^k\omega\|_{L^2\Lambda^k} \leq Ch^s\|\omega\|_{H^s\Lambda^k},$$

  for $\omega \in H^s\Lambda^k$ and $0 \leq s \leq r+1$. Moreover, for all $\omega \in L^2\Lambda^k$, $\pi_h^k\omega \to \omega$ in L^2 as $h \to 0$.
- *Let Λ_h^k be one of the spaces $\mathcal{P}_r\Lambda^k$ or $\mathcal{P}_r^-\Lambda^k$ with $r \geq 1$. Then,*

$$\|d(\omega - \pi_h^k\omega)\|_{L^2\Lambda^k} \leq Ch^s\|d\omega\|_{H^s\Lambda^k},$$

  for $\omega \in H^s\Lambda^k$ and $0 \leq s \leq r$.

We now consider the three-dimensional case. In this case, $\mathcal{P}_r\Lambda^0$ is the family of Lagrange elements of degree r; $\mathcal{P}_r^-\Lambda^1$ and $\mathcal{P}_r\Lambda^1$ are the family of Nédélec edge elements of the first and second kind (respectively) of degree r; $\mathcal{P}_r^-\Lambda^2$ and $\mathcal{P}_r\Lambda^2$ are the family of Nédélec face elements of the first and second kind (respectively) of degree r; and $\mathcal{P}_{r-1}\Lambda^3$ is the family of piecewise polynomials of degree $r-1$.

4.8 Scalar Wave Equation

We now start investigating examples of application of discretizations of the de Rham complex: we start with the acoustic scalar wave equation,

$$\ddot{u} - \Delta u = 0,$$

for a scalar field u, as a particular case of the theory developed. We consider the beginning of the de Rham complex,

$$0 \to H^1(\mathbb{R}) \xrightarrow{\text{grad}} L^2(\mathbb{V}),$$

with Hilbert spaces $W^{-1} = 0$, $W^0 = L^2(\mathbb{R})$, and $W^1 = L^2(\mathbb{V})$. The associated well-posed Hodge wave equation in weak form is given by seeking

$$(u, \rho) \in C^0([0,T], H^1(\mathbb{R}) \times L^2(\mathbb{V})) \cap C^1([0,T], L^2(\mathbb{R}) \times L^2(\mathbb{V}))$$

such that

$$(\dot{u}, v) = -(\rho, \operatorname{grad} v),$$
$$(\dot{\rho}, \mu) = (\operatorname{grad} u, \mu),$$

for any $v \in H^1(\mathbb{R})$ and $\mu \in L^2(\mathbb{V})$. Even though we wrote this problem in mixed form, the second equation immediately implies that $\dot{\rho} = \operatorname{grad} u$. Therefore, we can eliminate that second equation, and seek

$$u \in C^0([0,T], H^1(\mathbb{R})) \cap C^1([0,T], L^2(\mathbb{R}))$$

such that

$$(\ddot{u}, v) = -(\operatorname{grad} u, \operatorname{grad} v),$$

for any $v \in H^1(\mathbb{R})$. This leads to the standard non-mixed formulation of the acoustic wave equation, and is discussed in [20, 21]. The space used is simply $\mathcal{P}_r\Lambda^0$, the continuous Galerkin elements of degree r. In this previous case, the boundary conditions are natural, and are $\rho \cdot \boldsymbol{n} = \int_0^t \operatorname{grad} u \cdot \boldsymbol{n} = 0$. A variation with essential boundary conditions can be derived using

$$0 \to \mathring{H}^1(\mathbb{R}) \xrightarrow{\operatorname{grad}} L^2(\mathbb{V}),$$

with Hilbert spaces $W^{-1} = 0$, $W^0 = L^2(\mathbb{R})$, and $W^1 = L^2(\mathbb{V})$, In this case, the boundary conditions are $u = 0$.

We can turn to another formulation of the acoustic wave equation by considering,

$$H(\operatorname{div}, \mathbb{V}) \xrightarrow{\operatorname{div}} L^2(\mathbb{R}) \to 0,$$

with Hilbert spaces $W^2 = L^2(\mathbb{V})$, $W^3 = L^2(\mathbb{R})$, and $W^4 = 0$. The associated well-posed Hodge wave equation in weak form is given by seeking

$$(\sigma, u) \in C^0([0,T], H(\operatorname{div}, \mathbb{V}) \times L^2(\mathbb{R})) \cap C^1([0,T], L^2(\mathbb{V}) \times L^2(\mathbb{R}))$$

such that

$$(\dot{\sigma}, \tau) = -(u, \operatorname{div} \tau),$$
$$(\dot{u}, v) = (\operatorname{div} \sigma, v),$$

for any $\tau \in H(\mathrm{div}, \mathbb{V})$ and $v \in L^2(\mathbb{R})$. This mixed formulation is discussed in [22] and using FEEC in [23]. For the discretization, we can choose either of

$$\mathcal{P}_r^- \Lambda^2 \times \mathcal{P}_{r-1}\Lambda^3,$$
$$\mathcal{P}_r \Lambda^2 \times \mathcal{P}_{r-1}\Lambda^3,$$

namely either of Raviart-Thomas element of degree r with discontinuous finite elements of degree r, or Brezzi-Douglas-Marini elements of degree r with discontinuous finite elements of degree $r-1$. In this previous case, the boundary conditions are natural, and are $u = 0$. A variation with essential boundary conditions can be derived using

$$\mathring{H}(\mathrm{div}, \mathbb{V}) \xrightarrow{\mathrm{div}} L^2(\mathbb{R}) \to 0,$$

with Hilbert spaces $W^2 = L^2(\mathbb{V})$, $W^3 = L^2(\mathbb{R})$, and $W^4 = 0$. In this case, the boundary conditions are $\sigma \cdot \boldsymbol{n} = 0$.

4.9 Vector Wave Equation

Since linearized EB system is reminiscent of Maxwell's equations, and Maxwell's equations are a special case of the vector wave equation, we now investigate the vector wave equation,

$$\ddot{u} - \operatorname{grad} \operatorname{div} u + \operatorname{curl} \operatorname{curl} u = 0,$$

for a vector field u, We thus realize the vector wave equation on an interval $[0, T]$ restricting ourselves to a bounded three-dimensional contractible domain Ω. We do so by considering

$$H^1(\mathbb{R}) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}, \mathbb{V}) \xrightarrow{\mathrm{curl}} L^2(\mathbb{V}),$$

with Hilbert spaces $L^2(\mathbb{R})$, $L^2(\mathbb{V})$, and $L^2(\mathbb{V})$. The associated well-posed Hodge wave equation in weak form is given

$$(\sigma, u, \rho) \in C^0([0,T], H^1(\mathbb{R}) \times H(\mathrm{curl}, \mathbb{V}) \times L^2(\mathbb{V})) \cap C^1([0,T], L^2(\mathbb{R}) \times L^2(\mathbb{V}) \times L^2(\mathbb{V}))$$

such that

$$(\dot{\sigma}, \tau) = (u, \operatorname{grad} \tau), \tag{4.12a}$$
$$(\dot{u}, v) = -(\rho, \operatorname{curl} v) - (\operatorname{grad} \sigma, v), \tag{4.12b}$$
$$(\dot{\rho}, \mu) = (\operatorname{curl} u, \mu), \tag{4.12c}$$

for any $\tau \in H^1(\mathbb{R})$, $v \in H(\mathrm{curl}, \mathbb{V})$, and $\rho \in L^2(\mathbb{V})$. Again, we notice that using the last equation holds, we can eliminate the third unknown. Thus, a simplified formulation is to seek

$$(\sigma, u) \in C^0([0,T], H^1(\mathbb{R}) \times H(\mathrm{curl}, \mathbb{V})) \cap C^1([0,T], L^2(\mathbb{R}) \times L^2(\mathbb{V}))$$

such that

$$\begin{aligned}
(\dot{\sigma}, \tau) &= (u, \operatorname{grad} \tau), \\
(\ddot{u}, v) &= -(\operatorname{curl} u, \operatorname{curl} v) - (\operatorname{grad} \dot{\sigma}, v),
\end{aligned}$$

for any $\tau \in H^1(\mathbb{R})$, and $v \in H(\mathrm{curl}, \mathbb{V})$. The boundary conditions on this formulation are natural, and are $u \cdot \boldsymbol{n} = 0$ and $\rho \times \boldsymbol{n} = \int_0^t \operatorname{curl} u \times \boldsymbol{n} = 0$. We call these conditions *magnetic boundary conditions.* We can also get essential boundary conditions by considering instead

$$\mathring{H}^1(\mathbb{R}) \xrightarrow{\text{grad}} \mathring{H}(\mathrm{curl}, \mathbb{V}) \xrightarrow{\text{curl}} L^2(\mathbb{V}), \tag{4.13}$$

with Hilbert spaces $L^2(\mathbb{R})$, $L^2(\mathbb{V})$, and $L^2(\mathbb{V})$. This leads to $\sigma = 0$, and $u \times \boldsymbol{n} = 0$. We call these conditions *electric boundary conditions.* Now, possible discretizations for $V_h^0 \times V_h^1 \times V_h^2$ can be either

$$\begin{gathered}
\mathcal{P}_{r+1} \times \mathcal{P}_r\Lambda^1 \times \mathcal{P}_{r-1}\Lambda^2, \\
\mathcal{P}_{r+1} \times \mathcal{P}_r\Lambda^1 \times \mathcal{P}_r^-\Lambda^2, \\
\mathcal{P}_r \times \mathcal{P}_r^-\Lambda^1 \times \mathcal{P}_{r-1}\Lambda^2, \\
\mathcal{P}_r \times \mathcal{P}_r^-\Lambda^1 \times \mathcal{P}_r^-\Lambda^2.
\end{gathered}$$

If the initial conditions are also such that $\sigma(0) = 0$, and $u(0)$ and $\rho(0)$ are divergence-free (weakly and strongly, respectively), then σ remains zero, and u and ρ remain divergence-free for all time afterwards. In that case, $u = \boldsymbol{E}$ and $\rho = \boldsymbol{B}$ are the electric and magnetic fields and solve Maxwell's equations.

Proposition 4.16. *Suppose $\mathring{H}(\mathrm{curl}, \mathbb{V}) \cap H(\mathrm{div}, \mathbb{V}, 0)$ is dense in $H(\mathrm{div}, \mathbb{V}, 0)$, and that $H(\mathrm{curl}, \mathbb{V}) \cap H(\mathrm{div}, \mathbb{V}, 0)$ is dense in $H(\mathrm{div}, \mathbb{V}, 0)$. Suppose also we have a solution (σ, u, ρ) to (4.12). If the initial conditions are such that $\sigma_0 = 0$, and u_0 and ρ_0 are divergence-free, then σ remains zero, and u and ρ remain divergence-free for all time afterwards.*

Proof. We verify the hypothesis of Corollary 4.5 to show that the constraints are propagated. The weak formulation is equivalent to the strong formulation. We thus study the strong formulation and set

$$\mathcal{L} = \begin{pmatrix} 0 & -\operatorname{div} & 0 \\ -\operatorname{grad} & 0 & -\operatorname{curl} \\ 0 & \operatorname{curl} & 0 \end{pmatrix}$$

with domain

$$D(\mathcal{L}) = \mathring{H}^1(\mathbb{R}) \times (\mathring{H}(\operatorname{curl}, \mathbb{V}) \cap H(\operatorname{div}, \mathbb{V})) \times H(\operatorname{curl}, \mathbb{V}),$$

and

$$K = 0 \times H(\operatorname{div}, \mathbb{V}, 0) \times H(\operatorname{div}, \mathbb{V}, 0),$$

where $H(\operatorname{div}, \mathbb{V}, 0) = \{v \in H(\operatorname{div}, \mathbb{V}) \mid \operatorname{div} v = 0\}$. To have the first itemized hypothesis of Corollary 4.5, we observe the following. On one hand, we take $u \in \mathring{H}(\operatorname{curl}, \mathbb{V}) \cap H(\operatorname{div}, \mathbb{V})$, and use the Hodge decomposition to write $u = \operatorname{grad} \tau + \operatorname{curl} \mu$, for some $\tau \in \mathring{H}^1(\mathbb{R})$ and $\mu \in \operatorname{curl} H(\operatorname{curl}, \mathbb{V})$, since $L^2(\mathbb{V}) = \operatorname{grad} \mathring{H}^1(\mathbb{R}) \oplus \operatorname{curl} H(\operatorname{curl}, \mathbb{V})$, assuming the de Rham sequence is exact for the 1-forms. Now, $\operatorname{curl} \mu \in \mathring{H}(\operatorname{curl}, \mathbb{V})$ since $\operatorname{grad} \tau$ and u are in $\mathring{H}(\operatorname{curl}, \mathbb{V})$. Since $H(\operatorname{div}, \mathbb{V}, 0) = \operatorname{curl} H(\operatorname{curl}, \mathbb{V})$, we see that

$$P_{H(\operatorname{div}, \mathbb{V}, 0)} u = \operatorname{curl} \mu \in \mathring{H}(\operatorname{curl}, \mathbb{V}) \cap H(\operatorname{div}, \mathbb{V}).$$

Thus,

$$P_{H(\operatorname{div}, \mathbb{V}, 0)} \mathring{H}(\operatorname{curl}, \mathbb{V}) \cap H(\operatorname{div}, \mathbb{V}) \subset \mathring{H}(\operatorname{curl}, \mathbb{V}) \cap H(\operatorname{div}, \mathbb{V}).$$

On the other hand, $\rho \in H(\operatorname{curl}, \mathbb{V})$, and use the Hodge decomposition to write $\rho = \operatorname{grad} w + \operatorname{curl} v$, for some $w \in \operatorname{grad} H^1(\mathbb{R})$, $v \in \operatorname{curl} \mathring{H}(\operatorname{curl}, \mathbb{V})$, since $L^2(\mathbb{V}) = \operatorname{grad} H^1(\mathbb{R}) \oplus \operatorname{curl} \mathring{H}(\operatorname{curl}, \mathbb{V})$ (assuming the de Rham sequence is exact for the 1-forms). Now, $\operatorname{curl} v \in H(\operatorname{curl}, \mathbb{V})$ since $\operatorname{grad} w$ and ρ are in $H(\operatorname{curl}, \mathbb{V})$. Since $H(\operatorname{div}, \mathbb{V}, 0) = \operatorname{curl} H(\operatorname{curl}, \mathbb{V})$, we see that

$$P_{H(\operatorname{div}, \mathbb{V}, 0)} \rho = \operatorname{curl} v \in \mathring{H}(\operatorname{curl}, \mathbb{V}) \cap H(\operatorname{div}, \mathbb{V}),$$

and so

$$P_{H(\mathrm{div},\mathbb{V},0)}H(\mathrm{curl},\mathbb{V}) \subset \mathring{H}(\mathrm{curl},\mathbb{V}) \cap H(\mathrm{div},\mathbb{V}).$$

To have the second itemized hypothesis of Corollary 4.5, we conjecture that $\mathring{H}(\mathrm{curl},\mathbb{V}) \cap H(\mathrm{div},\mathbb{V},0)$ is dense in $H(\mathrm{div},\mathbb{V},0)$, and that $H(\mathrm{curl},\mathbb{V}) \cap H(\mathrm{div},\mathbb{V},0)$ is dense in $H(\mathrm{div},\mathbb{V},0)$. The third itemized hypothesis holds since

$$\begin{gathered}
-\operatorname{div}(\mathring{H}(\mathrm{curl},\mathbb{V}) \cap H(\mathrm{div},\mathbb{V},0)) = 0, \\
-\operatorname{grad}(\mathring{H}^1(\mathbb{R}) \cap 0) - \operatorname{curl}(H(\mathrm{curl},\mathbb{V}) \cap H(\mathrm{div},\mathbb{V},0)) \subset H(\mathrm{div},\mathbb{V},0), \\
\operatorname{curl}(\mathring{H}(\mathrm{curl},\mathbb{V}) \cap H(\mathrm{div},\mathbb{V})) \subset H(\mathrm{div},\mathbb{V},0).
\end{gathered}$$

Thus, by Corollary 4.5, the solution remains in K. □

Therefore, we can eliminate σ, and get another formulation for Maxwell's equations: seek

$$(u,\rho) \in C^0([0,T], H(\mathrm{curl},\mathbb{V}) \times L^2(\mathbb{V})) \cap C^1([0,T], L^2(\mathbb{V}) \times L^2(\mathbb{V}))$$

such that

$$\begin{aligned}
(\dot{u}, v) &= -(\rho, \operatorname{curl} v), \\
(\dot{\rho}, \mu) &= (\operatorname{curl} u, \mu),
\end{aligned}$$

for any $v \in H(\mathrm{curl},\mathbb{V})$ and $\rho \in L^2(\mathbb{V})$. We can again eliminate the last equation and seek

$$u \in C^0([0,T], H(\mathrm{curl},\mathbb{V})) \cap C^1([0,T], L^2(\mathbb{V}))$$

such that

$$(\ddot{u}, v) = -(\operatorname{curl} u, \operatorname{curl} v),$$

for any $v \in H(\mathrm{curl},\mathbb{V})$.

Other formulations of the vector wave equation (and of Maxwell's equations) can be obtained using the complex

$$H(\mathrm{curl},\mathbb{V}) \xrightarrow{\mathrm{curl}} H(\mathrm{div},\mathbb{V}) \xrightarrow{\mathrm{div}} L^2(\mathbb{R}),$$

with Hilbert spaces $L^2(\mathbb{V})$, $L^2(\mathbb{V})$, and $L^2(\mathbb{R})$, and considering the associated well-posed Hodge wave equation. Moreover, we can consider the complex with essential boundary conditions. In this approach, the primary variable is a 2-form (the magnetic field $\boldsymbol{B}$ for Maxwell's equations), in contrast to the approach discussed in which a 1-form (the electric field $\boldsymbol{E}$) is the primary variable. We will not pursue the 2-form approach here.

4.10 Application to the Linearized EB System

Having studied the vector wave equation, we are now ready to investigate the linearized EB system. We thus realize the EB system as a Hodge wave equation. To do so, we introduce a new variable, $\sigma(t) = -\int_0^t \operatorname{div} \boldsymbol{E}$, which will vanish for the exact solution for the EB system. We consider an interval $[0,T]$ and restrict the problem to a bounded three-dimensional contractible domain, where we consider the complex,

$$\mathring{H}^1(\mathbb{V}) \xrightarrow{\text{grad}} \mathring{H}(\text{curl}, \mathbb{M}) \xrightarrow{\text{curl}} L^2(\mathbb{M}),$$

with Hilbert spaces $L^2(\mathbb{V})$, $L^2(\mathbb{M})$, and $L^2(\mathbb{M})$. Note that this complex may be obtained from (4.13) by tensoring with $\mathbb{V}$, i.e. it is essentially the product of three copies of (4.13). In this case, the related strong form of the mixed abstract Hodge wave equation seeks

$$\begin{aligned}
\sigma &\in C^0([0,T], \mathring{H}^1(\mathbb{V})) \cap C^1([0,T], L^2(\mathbb{V})),\\
\boldsymbol{E} = u &\in C^0([0,T], \mathring{H}(\text{curl}, \mathbb{M}) \cap H(\text{div}, \mathbb{M})) \cap C^1([0,T], L^2(\mathbb{M})),\\
\boldsymbol{B} = \rho &\in C^0([0,T], H(\text{curl}, \mathbb{M})) \cap C^1([0,T], L^2(\mathbb{M})),
\end{aligned}$$

satisfying

$$\dot{\sigma} = -\operatorname{div} \boldsymbol{E}, \tag{4.14a}$$

$$\dot{\boldsymbol{E}} = -\operatorname{grad} \sigma - \operatorname{curl} \boldsymbol{B}, \tag{4.14b}$$

$$\dot{\boldsymbol{B}} = \operatorname{curl} \boldsymbol{E}, \tag{4.14c}$$

where the initial conditions $(\sigma_0, \boldsymbol{E}_0, \boldsymbol{B}_0) \in H^1(\mathbb{V}) \times \mathring{H}(\text{curl}, \mathbb{M}) \cap H(\text{div}, \mathbb{M}) \times H(\text{curl}, \mathbb{M})$. We call this system the *matrix wave equation.* Using the theory developed for the Hodge wave equation, we can now find discretizations $V_h^0 \times V_h^1 \times V_h^2$ for this formulations,

$$\mathcal{P}_{r+1}\Lambda^0 \otimes \mathbb{V} \times \mathcal{P}_r\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_{r-1}\Lambda^2 \otimes \mathbb{V},$$
$$\mathcal{P}_{r+1}\Lambda^0 \otimes \mathbb{V} \times \mathcal{P}_r\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_r^-\Lambda^2 \otimes \mathbb{V},$$
$$\mathcal{P}_{r+1}\Lambda^0 \otimes \mathbb{V} \times \mathcal{P}_r^-\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_{r-1}\Lambda^2 \otimes \mathbb{V},$$
$$\mathcal{P}_{r+1}\Lambda^0 \otimes \mathbb{V} \times \mathcal{P}_r^-\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_r^-\Lambda^2 \otimes \mathbb{V}.$$

Notice that, in this approach, $\boldsymbol{E}$ and $\boldsymbol{B}$ are sought as matrix fields which are not necessarily symmetric nor trace-free nor divergence-free. However, we now show that all three conditions hold for the solution, as long as the initial data is selected appropriately. Moreover, in this case, the linearized EB evolution equations in Proposition 3.4 hold.

Proposition 4.17. *Suppose that*

$$P_{L^2(\mathrm{TSD})}(\mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M})) \subset \mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M}),$$
$$P_{L^2(\mathrm{TSD})}(H(\mathrm{curl}, \mathbb{M})) \subset H(\mathrm{curl}, \mathbb{M}),$$

$\mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M}, 0)$ *is dense in* $H(\mathrm{div}, \mathbb{M}, 0)$*, and* $H(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M}, 0)$ *is dense in* $H(\mathrm{div}, \mathbb{M}, 0)$*. Given initial data*

$$(\sigma_0, \boldsymbol{E}_0, \boldsymbol{B}_0) \in \mathring{H}^1(\mathbb{V}) \times \mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M}) \times H(\mathrm{curl}, \mathbb{M})$$

such that $\sigma_0 = 0$*, and* $\boldsymbol{E}_0$ *and* $\boldsymbol{B}_0$ *are TSD, let* $(\sigma, \boldsymbol{E}, \boldsymbol{B})$ *be the unique solution of the equations* (4.14)*. Then, for all time,* $\sigma = 0$*, and* $\boldsymbol{E}$ *and* $\boldsymbol{B}$ *are TSD, and the evolution equations in Proposition 3.4 hold.*

Proof. Let K be the closed subspace of $\boldsymbol{W}$ of $(0, \boldsymbol{F}, \boldsymbol{G})$ such that $\boldsymbol{F}$ and $\boldsymbol{G}$ are TSD. We let $L^2(\mathrm{TDS})$ be the subspace of $L^2(\mathbb{M})$ such that the matrix fields are TSD.

We verify the hypothesis of Theorem 4.5 to show that the constraints are propagated. We set

$$\mathcal{L} = \begin{pmatrix} 0 & -\mathrm{div} & 0 \\ -\mathrm{grad} & 0 & -\mathrm{curl} \\ 0 & \mathrm{curl} & 0 \end{pmatrix}$$

with domain

$$D(\mathcal{L}) = \mathring{H}^1(\mathbb{V}) \times (\mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M})) \times H(\mathrm{curl}, \mathbb{M}).$$

To have the first itemized hypothesis of Theorem 4.5, we conjecture that

$$P_{L^2(\mathrm{TSD})}(\mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M})) \subset \mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M}),$$
$$P_{L^2(\mathrm{TSD})}(H(\mathrm{curl}, \mathbb{M})) \subset H(\mathrm{curl}, \mathbb{M}).$$

Similarly, to have the second itemized hypothesis of Theorem 4.5, as done for the scalar wave equation, we conjecture that $\mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M}, 0)$ is dense in $H(\mathrm{div}, \mathbb{M}, 0)$, and that $H(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M}, 0)$ is dense in $H(\mathrm{div}, \mathbb{M}, 0)$, so that $\mathring{H}(\mathrm{curl}, \mathbb{M}) \cap L^2(\mathrm{TSD})$ is dense in $L^2(\mathrm{TSD})$, and that $H(\mathrm{curl}, \mathbb{M}) \cap L^2(\mathrm{TSD})$ is dense in $L^2(\mathrm{TSD})$. Finally, the third itemized hypothesis holds since

$$-\,\mathrm{div}(\mathring{H}(\mathrm{curl}, \mathbb{V}) \cap L^2(\mathrm{TSD})) = 0,$$
$$-\,\mathrm{grad}(\mathring{H}^1(\mathbb{V}) \cap 0) - \mathrm{curl}(H(\mathrm{curl}, \mathbb{M}) \cap L^2(\mathrm{TSD})) \subset L^2(\mathrm{TSD}),$$
$$\mathrm{curl}(\mathring{H}(\mathrm{curl}, \mathbb{V}) \cap H(\mathrm{div}, \mathbb{V})) \subset L^2(\mathrm{TSD}),$$

using Lemma 3.5. Therefore, by Theorem 4.5, $\sigma = 0$, and $\boldsymbol{E}$ and $\boldsymbol{B}$ are TSD, for all time. This concludes the proof. □

The fact that $\sigma = 0$ remains true for divergence-free initial data for $\boldsymbol{E}$ suggests that we can consider a simplified system: seek

$$\boldsymbol{E} \in C^0([0,T], \mathring{H}(\mathrm{curl}, \mathbb{M}) \cap H(\mathrm{div}, \mathbb{M})) \cap C^1([0,T], L^2(\mathbb{M})),$$
$$\boldsymbol{B} \in C^0([0,T], H(\mathrm{curl}, \mathbb{M})) \cap C^1([0,T], L^2(\mathbb{M})),$$

such that

$$\dot{\boldsymbol{E}} = -\,\mathrm{curl}\,\boldsymbol{B}, \tag{4.15a}$$
$$\dot{\boldsymbol{B}} = \mathrm{curl}\,\boldsymbol{E}, \tag{4.15b}$$

with intial data satisfying $\mathrm{div}\,\boldsymbol{E}_0 = 0$. Theses are the evolution equations found in Proposition 3.4. We have already shown in Proposition 3.6 that, if a solution exists with appropriate initial data, the constraints are propagated. We now show that a solution does indeed exist.

Proposition 4.18. *The systems* (4.14) *and* (4.15) *are equivalent in the following sense. Suppose we are given initial conditions* $(\boldsymbol{E}_0, \boldsymbol{B}_0) \in V^1 \times V^2$ *to* (4.15) *such that* $\mathrm{div}\,\boldsymbol{E}_0 = 0$.

- *If* $(\sigma, \boldsymbol{E}, \boldsymbol{B})$ *is the solution to* (4.14) *with initial conditions* $(0, \boldsymbol{E}_0, \boldsymbol{B}_0)$*, then* $\sigma = 0$ *and* $(\boldsymbol{E}, \boldsymbol{B})$ *is the solution to* (4.15) *with initial conditions* $(\boldsymbol{E}_0, \boldsymbol{B}_0)$*.*
- *If* $(\boldsymbol{E}, \boldsymbol{B})$ *is the solution to* (4.15) *with initial conditions* $(\boldsymbol{E}_0, \boldsymbol{B}_0)$*, then* $(0, \boldsymbol{E}, \boldsymbol{B})$ *is the solution to* (4.14) *with initial conditions* $(0, \boldsymbol{E}_0, \boldsymbol{B}_0)$*.*

Using the theory developed for the Hodge wave equation, we can now find discretizations $V_h^1 \times V_h^2$ for this formulations,

$$\begin{gathered}
\mathcal{P}_r\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_{r-1}\Lambda^2 \otimes \mathbb{V}, \\
\mathcal{P}_r\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_r^-\Lambda^2 \otimes \mathbb{V}, \\
\mathcal{P}_r^-\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_{r-1}\Lambda^2 \otimes \mathbb{V}, \\
\mathcal{P}_r^-\Lambda^1 \otimes \mathbb{V} \times \mathcal{P}_r^-\Lambda^2 \otimes \mathbb{V}.
\end{gathered}$$

This is thus a first formulation of the linearized EB system that we can implement. We show a second possibility in the next section.

4.11 Another Complex for the Linearized EB System

We now introduce a formulation of the linearized EB system in which the symmetry of the electric part is strongly imposed. By enforcing this symmetry, the hope is that the system will be able to satisfy the constraints more easily whenever the system is modified with coefficients or lower order terms. In order to realize this formulation as a Hodge wave equation, we introduce the new variable $\sigma(t) = \int_0^t \operatorname{div}\operatorname{div} \boldsymbol{E}$. We then consider the *second order complex with strong symmetries*,

$$\mathring{H}^2(\mathbb{R}) \xrightarrow{\text{grad grad}} \mathring{H}(\text{curl}, \mathbb{S}) \xrightarrow{\text{curl}} L^2(\mathbb{T}), \tag{4.16}$$

with adjoints

$$L^2(\mathbb{R}) \xleftarrow{\text{div div}} H(\text{div div}, \mathbb{S}) \xleftarrow{\text{sym curl}} H(\text{sym curl}, \mathbb{T}), \tag{4.17}$$

where

$$\begin{aligned}
H(\text{div div}, \mathbb{S}) &= \left\{ u \in L^2(\mathbb{S}) \mid \operatorname{div}\operatorname{div} u \in L^2(\mathbb{R}) \right\}, \\
H(\text{sym curl}, \mathbb{T}) &= \left\{ u \in L^2(\mathbb{T}) \mid \operatorname{sym}\operatorname{curl} u \in L^2(\mathbb{S}) \right\},
\end{aligned}$$

The Hilbert spaces are $W^0 = L^2(\mathbb{R})$, $W^1 = L^2(\mathbb{S})$, and $W^2 = L^2(\mathbb{T})$, where $L^2(\mathbb{S})$ is the subspace of $L^2(\mathbb{M})$ with symmetric matrices, and $L^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{M})$ with traceless matrices. If we show that these complexes are closed, we have that the associated Hodge wave have an unique solution:

$$\begin{aligned}\sigma &\in C^0([0,T], \mathring{H}^2(\mathbb{R})) \cap C^1([0,T], L^2(\mathbb{R})),\\ \boldsymbol{E} = u &\in C^0([0,T], \mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S})) \cap C^1([0,T], L^2(\mathbb{M})),\\ \boldsymbol{B} = \rho &\in C^0([0,T], H(\operatorname{sym}\operatorname{curl}, \mathbb{T})) \cap C^1([0,T], L^2(\mathbb{T})),\end{aligned}$$

such that

$$\dot{\sigma} = \operatorname{div}\operatorname{div}\boldsymbol{E}, \tag{4.18a}$$

$$\dot{\boldsymbol{E}} = -\operatorname{grad}\operatorname{grad}\sigma - \operatorname{sym}\operatorname{curl}\boldsymbol{B}, \tag{4.18b}$$

$$\dot{\boldsymbol{B}} = \operatorname{curl}\boldsymbol{E}, \tag{4.18c}$$

where the initial conditions

$$(\sigma, \boldsymbol{E}, \boldsymbol{B})(0) \in \mathring{H}^2(\mathbb{R}) \times \mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S}) \times H(\operatorname{sym}\operatorname{curl}, \mathbb{T}).$$

We show in the next chapter that these complexes are closed.

We are interested in the case when the initial conditions are such that $\sigma(0) = 0$, and $\boldsymbol{E}(0)$ and $\boldsymbol{B}(0)$ are TSD. We show that $\sigma = 0$, and that $\boldsymbol{E}$ and $\boldsymbol{B}$ are TSD for all time.

Proposition 4.19. *Suppose that*

$$P_{L^2(\mathrm{TSD})}(\mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S})) \subset \mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S}),$$
$$P_{L^2(\mathrm{TSD})}(H(\operatorname{sym}\operatorname{curl}, \mathbb{T})) \subset H(\operatorname{sym}\operatorname{curl}, \mathbb{T}),$$

$\mathring{H}(\operatorname{curl}, \mathbb{S}) \cap L^2(\mathrm{TSD})$ *is dense in* $L^2(\mathrm{TSD})$*, and* $H(\operatorname{sym}\operatorname{curl}, \mathbb{T}) \cap L^2(\mathrm{TSD})$ *is dense in* $L^2(\mathrm{TSD})$*. Given initial conditions*

$$(\sigma_0, \boldsymbol{E}_0, \boldsymbol{B}_0) \in H^2 \times H(\operatorname{curl}, \mathbb{S}) \cap \mathring{H}(\operatorname{div}\operatorname{div}, \mathbb{S}) \times \mathring{H}(\operatorname{sym}\operatorname{curl}, \mathbb{T})$$

such that $\sigma_0 = 0$*, and that* $\boldsymbol{E}_0$ *and* $\boldsymbol{B}_0$ *are TSD, and we let* $(\sigma, \boldsymbol{E}, \boldsymbol{B})$ *be the unique solution of the equations* (4.18)*. Then, for all time,* $\sigma = 0$*, and* $\boldsymbol{E}$ *and* $\boldsymbol{B}$ *are TSD.*

Proof. Let K be the closed subspace of $\boldsymbol{W}$ of $(0, \boldsymbol{F}, \boldsymbol{G})$ such that $\boldsymbol{F}$ and $\boldsymbol{G}$ are TSD. We verify the hypothesis of Theorem 4.5 to show that the constraints are propagated. We set

$$\mathcal{L} = \begin{pmatrix} 0 & \operatorname{div}\operatorname{div} & 0 \\ -\operatorname{grad}\operatorname{grad} & 0 & -\operatorname{sym}\operatorname{curl} \\ 0 & \operatorname{curl} & 0 \end{pmatrix}$$

with domain

$$D(\mathcal{L}) = \mathring{H}^2(\mathbb{R}) \times \mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S}) \times H(\operatorname{sym}\operatorname{curl}, \mathbb{T}).$$

To have the first itemized hypothesis of Theorem 4.5, we conjecture that

$$P_{L^2(\mathrm{TSD})}(\mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S})) \subset \mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S}),$$
$$P_{L^2(\mathrm{TSD})}(H(\operatorname{sym}\operatorname{curl}, \mathbb{T})) \subset H(\operatorname{sym}\operatorname{curl}, \mathbb{T}).$$

To also have the second itemized hypothesis of Theorem 4.5, we conjecture that $\mathring{H}(\operatorname{curl}, \mathbb{S}) \cap L^2(\mathrm{TSD})$ is dense in $L^2(\mathrm{TSD})$, and that $H(\operatorname{sym}\operatorname{curl}, \mathbb{T}) \cap L^2(\mathrm{TSD})$ is dense in $L^2(\mathrm{TSD})$. Finally, the third itemized hypothesis holds since

$$\operatorname{div}\operatorname{div}(\mathring{H}(\operatorname{curl}, \mathbb{S}) \cap L^2(\mathrm{TSD})) = 0,$$
$$-\operatorname{grad}(\mathring{H}^2(\mathbb{R}) \cap 0) - \operatorname{sym}\operatorname{curl}(H(\operatorname{sym}\operatorname{curl}, \mathbb{T}) \cap L^2(\mathrm{TSD})) \subset L^2(\mathrm{TSD}),$$
$$\operatorname{curl}(\mathring{H}(\operatorname{curl}, \mathbb{S}) \cap H(\operatorname{div}\operatorname{div}, \mathbb{S}) \cap L^2(\mathrm{TSD})) \subset L^2(\mathrm{TSD}),$$

using Lemma 3.5. Therefore, by Theorem 4.5, $\sigma = 0$, and $\boldsymbol{E}$ and $\boldsymbol{B}$ are TSD, for all time. This concludes the proof. □

Since the symmetry on $\boldsymbol{E}$ is imposed strongly at the continuous level, the finite elements would also need to have symmetry imposed strongly at the discrete level. Moreover, an appropriate discretization of $\mathring{H}^2(\mathbb{R})$ is needed. As we will identify a related complex that is much simpler to discretize in the next chapter, we do not discretize this complex directly. In addition to the first formulation found in the previous section, this next complex will then give us a second formulation of the linearized EB system

Chapter 5

Time-Independent BGG Construction

We introduced a formulation of the linearized EB system in which the symmetry of the electric part is strongly imposed, using the second order complex (4.16) with strong symmetries. However, finding finite elements with second derivatives and strong symmetry lead to higher number of degrees of freedom. To avoid this issue, we borrow ideas from plate bending for $\mathring{H}^2(\mathbb{R})$ and elasticity for $\mathring{H}(\mathrm{curl}, \mathbb{S})$. The first idea is to replace $\mathring{H}^2(\mathbb{R})$ by the two spaces $\mathring{H}^1(\mathbb{R}) \times \mathring{H}^1(\mathbb{V})$ using a multiplier, as in [24, 25] for plate bending. The second idea is to impose the symmetry weakly, as done in [26]. Combining these two ideas results in a new complex which we analyze through a new abstract framework. This framework enables us to build complexes out of others.

We first develop this framework in the time-independent context, and so we begin by considering a time-independent version of the EB system by looking at the mixed Hodge Laplacian, which is well-posed, of the complex (4.16): $\sigma \in \mathring{H}^2(\mathbb{R})$, $\boldsymbol{E} \in \mathring{H}(\mathrm{curl}, \mathbb{S}) \cap H(\mathrm{div}\,\mathrm{div}, \mathbb{S})$, and $\boldsymbol{B} \in H(\mathrm{sym}\,\mathrm{curl}, \mathbb{T})$, such that

$$\sigma = \mathrm{div}\,\mathrm{div}\,\boldsymbol{E},$$

$$\mathrm{grad}\,\mathrm{grad}\,\sigma + \mathrm{sym}\,\mathrm{curl}\,\boldsymbol{B} = 0,$$

$$\boldsymbol{B} = \mathrm{curl}\,\boldsymbol{E}.$$

This is the strong form of the time-independent EB system with strong symmetries. We shall identify finite element spaces for the weak form of the weak symmetries formulation of this system.

In the first section, we introduce an abstract framework for the construction of new complexes from previous ones, and study a well-posed associated problem. In the second section, we discuss the discretization of this problem. In the last section, we then apply this theory to the version of the time-independent EB system just mentioned.

5.1 Abstract Framework

In this section, we develop an abstract framework to allow us to build new complexes from previous ones, and then study the associated Hodge Laplacian. We suppose that we are given Hilbert spaces W^0, W^1, W^2, $\widetilde{W}^0$, $\widetilde{W}^1$, and $\widetilde{W}^2$, along with closed, densely defined, unbounded, and closed range, operators $d^0 : V^0 \to V^1$, $d^1 : V^1 \to W^2$, $d^0 : \widetilde{V}^0 \to \widetilde{V}^1$, and $d^1 : \widetilde{V}^1 \to \widetilde{W}^2$, and domains V^0, V^1, $\widetilde{V}^0$, and $\widetilde{V}^1$, in their respective Hilbert spaces, equipped with the graph norm. We suppose that $d \circ d = 0$, and thus have two closed Hilbert complexes, which we further assume exact. We finally assume that we have bounded linear maps $S_0 : \widetilde{W}^0 \to W^1$, and $S_1 : \widetilde{W}^1 \to W^2$ such that S_0 is injective, $S_0\widetilde{V}^0 \subset V^1$, and $d^1 S_0 = -S_1 d^0$ on $\widetilde{V}^0$,

$$
\begin{tikzcd}
0 \arrow[r, hook] & V^0 \arrow[r, "d"] & V^1 \arrow[r, "d"] & W^2 \\
0 \arrow[r, hook] & \widetilde{V}^0 \arrow[r, "d"] \arrow[ur, "S"] & \widetilde{V}^1 \arrow[r, "d"] \arrow[ur, "S"] & \widetilde{W}^2
\end{tikzcd}
$$

to relate the two complexes. Moreover, we assume a *regularity property*: $dS\widetilde{V}^0 = dV^1$. This allows us to consider *first order complex with weak symmetries*,

$$\Gamma \xrightarrow{(0\ d)} \widetilde{V}^1 \xrightarrow{\binom{S}{d}} W^2 \times \widetilde{W}^2, \tag{5.1}$$

where

$$\Gamma = \left\{(\sigma, \phi) \in V^0 \times \widetilde{V}^0 \mid d\sigma = S\phi\right\} = \left\{(\sigma, S^{-1}d\sigma) \mid \sigma \in V^0, d\sigma \in S\widetilde{V}^0\right\}.$$

The associated Hilbert spaces are $\bar{\Gamma}$, $\widetilde{W}^1$, and $W^2 \times \widetilde{W}^2$, where $\bar{\Gamma}$ is the completion of Γ in the norm $\|(\sigma, \phi)\| = \|\sigma\|$. The injectivity of S_0 is used here for defining the norm for $\bar{\Gamma}$. This complex inherits some properties from the other two Hilbert complexes.

Proposition 5.1. *Suppose we have two exact closed Hilbert complexes, namely V^i and $\widetilde{V}^i$ with respective Hilbert spaces W^i and $\widetilde{W}^i$, for $0 \leq i \leq 2$. Suppose also we have bounded linear maps $S_0 : \widetilde{W}^0 \to W^1$, and $S_1 : \widetilde{W}^1 \to W^2$ such that S_0 is injective, $S_0\widetilde{V}^0 \subset V^1$, $d^1 S_0 = -S_1 d^0$ on $\widetilde{V}^0$, and the regularity property holds. The first order complex with weak symmetries* (5.1) *is a closed Hilbert complex, with associated Hilbert spaces $\bar{\Gamma}$, $\widetilde{W}^1$, and $W^2 \times \widetilde{W}^2$.*

Proof. We show that the operator $\begin{pmatrix} 0 & d \end{pmatrix}$ is closed. Consider a sequence $(\sigma_n, \phi_n) \in \Gamma$ converging to $(\sigma, \phi) \in \bar{\Gamma}$ in $\bar{\Gamma}$ such that $d\phi_n$ converges in $\widetilde{W}^1$ to $u \in \widetilde{W}^1$. We want to show that $(\sigma, \phi) \in \Gamma$ and $d\phi = u$. Since $d\phi_n$ converges in $\widetilde{W}^1$ to $u \in \widetilde{W}^1$ and d has closed range, we have $u = d\phi_0$, for some $\phi_0 \in \widetilde{V}^0$. Since the bottom sequence is exact at $\widetilde{V}^0$, we have that ϕ_n converges in $\widetilde{V}^0$ to $\phi_0 = \phi$. Moreover, we know that $d\sigma_n = S\phi_n$ which converges to $S\phi$ in W^1. However, since $(\sigma_n, \phi_n) \in \Gamma$ converges to (σ, ϕ) in $\bar{\Gamma}$, we know that σ_n converges in W^0 to $\sigma \in W^0$. Therefore, since d is a closed unbounded operator, $d\sigma = S\phi$, and $(\sigma, \phi) \in \Gamma$, as desired.

We show that

$$\begin{pmatrix} 0 & d \end{pmatrix}$$

has closed range. Consider a sequence

$$\begin{pmatrix} 0 & d \end{pmatrix} \begin{pmatrix} \sigma_n \\ \phi_n \end{pmatrix}$$

converging to u in $\widetilde{W}^1$, with $(\sigma_n, \phi_n) \in \Gamma$ for all n. Hence, we have $d\phi_n$ converging to u in $\widetilde{W}^1$. Since d has closed range, $u = d\phi$, for some $\phi \in \widetilde{V}^0$. Since the bottom complex is exact at $\widetilde{V}^0$, ϕ_n converges to ϕ in $\widetilde{W}^0$. However, we have $d\sigma_n = S\phi_n$ which converges to $S\phi$ in W^1, since S is continuous. Thus, $d\sigma_n$ converges in W^1, but, since d has closed range, $d\sigma_n$ converges to $d\sigma$ in W^1, for some $\sigma \in V^0$. Since the top complex is also exact at V^0, σ_n converges to σ in W^0. We note that $d\sigma = S\phi$, and

$$\begin{pmatrix} 0 & d \end{pmatrix} \begin{pmatrix} \sigma_n \\ \phi_n \end{pmatrix}$$

converges to

$$u = \begin{pmatrix} 0 & d \end{pmatrix} \begin{pmatrix} \sigma \\ \phi \end{pmatrix}$$

in $\widetilde{W}^1$, as desired.

We now note that the operator

$$\begin{pmatrix} S \\ d \end{pmatrix}$$

is clearly closed, and show that it has closed range. We take a sequence

$$\begin{pmatrix} S \\ d \end{pmatrix} u_n \to \begin{pmatrix} \rho \\ \mu \end{pmatrix}$$

in $W^2 \times \widetilde{W}^2$, with $u_n \in \widetilde{V}^1$. Since the bottom complex has a Hodge decomposition and is exact at $\widetilde{V}^1$, we have $u_n = d\phi_n + w_n$, for some $\phi_n \in \widetilde{V}^0$ and $w_n \in \widetilde{\mathfrak{Z}}^{\perp_{\widetilde{V}}}$, where $\widetilde{\mathfrak{Z}}$ is the null space of d^0. Since $dw_n = du_n$ converges to μ in $\widetilde{W}^2$ and $w_n \in \widetilde{\mathfrak{Z}}^{\perp_{\widetilde{V}}}$, we have that, for some $w \in \widetilde{V}^1$, w_n converges to w in $\widetilde{W}^1$ and $\mu = dw$. Now, we note that $-dS\phi_n = Sd\phi_n = Su_n - Sw_n$ converges to $\rho - Sw$ in W^2. However, d^1 has closed range, so $-dS\phi_n$ converges in W^1 to $-d\lambda$ for some $\lambda \in V^1$, and $-d\lambda = \rho - Sw$. Now, by the assumed regularity property $dV^1 = dS\widetilde{V}^0$, there exists $\phi \in \widetilde{V}^0$ such that $-dS\phi = -d\lambda = \rho - Sw$. Hence, we set $u = d\phi + w \in \widetilde{V}^1$, and see that $\rho = Su$ and $\mu = du$, as desired.

Finally, we need to show that

$$0 = \begin{pmatrix} S \\ d \end{pmatrix} \begin{pmatrix} 0 & d \end{pmatrix} \Gamma.$$

Thus, given $(\sigma, \phi) \in \Gamma$, we need to show that $Sd\phi = 0$. However, this is true, since $Sd\phi = -dS\phi = -d(d\sigma) = 0$. This concludes the proof. $\square$

We note that, since we have the closed Hilbert complex (5.1), we have the following Poincaré inequalities. For any $(\sigma, \phi) \in \Gamma$,

$$\|\sigma\| \leq C_p \|d\phi\|,$$

and, for any $u \in \widetilde{V}^1$ such that $u \perp \mathcal{N}(d) \cap \mathcal{N}(S) \cap \widetilde{V}^1$,

$$\|u\| \leq C_p \left(\|S_1 u\| + \|du\| \right).$$

The associated well-posed Hodge Laplacian problem is to find $(\sigma, \phi) \in \Gamma$ and $u \in \widetilde{V}^1$ such that

$$(\sigma, \tau) - (u, d\psi) = 0, \qquad (\tau, \psi) \in \Gamma, \tag{5.2a}$$

$$(d\phi, v) + (Su, Sv) + (du, dv) = (f, v), \qquad v \in \widetilde{V}^1, \tag{5.2b}$$

with $f \in \widetilde{W}^1$. To impose the constraint used to define Γ, we introduce a Lagrange multiplier $\lambda \in W^1$, and so the modified problem is to find $(\sigma, \phi) \in V^0 \times \widetilde{V}^0$, $u \in \widetilde{V}^1$, and $\lambda \in W^1$ such that

$$(\sigma, \tau) - (u, d\psi) - (\lambda, d\tau - S\psi) = 0, \qquad (\tau, \psi) \in V^0 \times \widetilde{V}^0, \tag{5.3a}$$

$$(d\phi, v) + (Su, Sv) + (du, dv) = (f, v), \qquad v \in \widetilde{V}^1, \tag{5.3b}$$

$$(d\sigma - S\phi, \mu) = 0, \qquad \mu \in W^1, \tag{5.3c}$$

with $f \in \widetilde{W}^1$. We let $a : (V^0 \times \widetilde{V}^0 \times \widetilde{V}^1) \times (V^0 \times \widetilde{V}^0 \times \widetilde{V}^1) \to \mathbb{R}$ be defined by

$$a(\sigma, \phi, u; \tau, \psi, v) = (\sigma, \tau) - (u, d\psi) + (d\phi, v) + (Su, Sv) + (du, dv),$$

and $b : (V^0 \times \widetilde{V}^0 \times \widetilde{V}^1) \times W^1 \to \mathbb{R}$ be defined by $b(\tau, \psi, v; \lambda) = -(\lambda, d\tau - S\psi)$.

Clearly, a solution $(\sigma, \phi, u, \lambda)$ to the system (5.3) is a solution (σ, ϕ, u) to the system (5.2) without Lagrange multiplier. If we show that the second system (5.3) is well-posed, we would have that a solution (σ, ϕ, u) to the first system is also a (unique) solution $(\sigma, \phi, u, \lambda)$ to the second one, for some unique λ.

We now suppose $S\widetilde{V}^0$ is dense in W^1, so that the seminorm

$$|||\lambda||| := \sup_{(\tau,\psi) \in V^0 \times \widetilde{V}^0} \frac{(\lambda, d\tau - S\psi)}{\|\tau\|_V + \|\psi\|_V},$$

is actually a norm. We denote by $\overline{W}^1$ the completion of W^1 with this norm. This is analogous to the derivation of error estimates for the Reissner–Mindlin equations in the limiting case $t = 0$ [25].

Proposition 5.2. *Suppose the hypothesis of Proposition 5.1 holds. The problem* (5.3) *is well-posed over $V^0 \times \widetilde{V}^0 \times \widetilde{V}^1 \times \overline{W}^1$, where $\overline{W}^1$ the completion of W^1 with the norm $|||\cdot|||$ just defined.*

Proof. We use Brezzi's Theorem 4.11. The kernel Z of b is $\Gamma \times \widetilde{V}^1$. First, we note that the inf-sup condition for b holds by definition of the norm on $\overline{W}^1$.

Second, we consider any $(\sigma, \phi) \in \Gamma$ and $u \in \widetilde{V}^1$. Since $u \in \widetilde{V}^1$, we can use a Hodge decomposition with the complex (5.1), so that $u = d\chi + w$, where $\chi \in \widetilde{V}^0$, $S\chi = d\rho$, for some $\rho \in V^0$, and $w \in (\mathfrak{Z}^1)^{\perp_V}$, with $\mathfrak{Z}^1$ denoting the null space of the operator $\left(\begin{smallmatrix} S \\ d \end{smallmatrix}\right)$. We can now fix a small $\epsilon > 0$, and take $(\tau, \psi, v) = (\sigma - \epsilon\rho, \phi - \epsilon\chi, d\phi + d\chi + \epsilon w) \in Z$, since $d(\sigma - \epsilon\rho) = S(\phi - \epsilon\chi)$, to show the inf-sup conditions for a over $Z \times Z$ holds. Indeed, this choice yields

$$
\begin{aligned}
&a(\sigma, \phi, u; \tau, \psi, v) \\
&= (\sigma, \tau) - (u, d\psi) + (d\phi, v) + (Su, Sv) + (du, dv) \\
&= (\sigma, \tau) - (d\chi, d\psi) - (w, d\psi) + (d\phi, v) + (Sd\chi, Sv) + (Sw, Sv) + (dw, dv) \\
&= (\sigma, \tau) - (d\chi, d\psi) + (d\phi, v) + (Sw, Sv) + (dw, dv),
\end{aligned}
$$

as $w \in (\mathfrak{Z}^1)^{\perp_V}$ and $d\psi \in \mathfrak{Z}^1$, since $Sd\phi = -dS\phi = -d(d\sigma) = 0$, and similarly $Sd\chi = 0$. Now, substituting τ, ψ, and v,

$$
\begin{aligned}
a(\sigma, \phi, u; \tau, \psi, v) &= \|\sigma\|^2 - \epsilon(\sigma, \rho) - (d\chi, d\phi) + \epsilon\|d\chi\|^2 \\
&\quad + \|d\phi\|^2 + (d\phi, d\chi) + \epsilon(d\phi, w) + (Sw, Sd\phi) + (Sw, Sd\chi) + \epsilon\|Sw\|^2 \\
&\quad + \epsilon\|dw\|^2 \\
&= \|\sigma\|^2 - \epsilon(\sigma, \rho) + \epsilon\|d\chi\|^2 + \|d\phi\|^2 + \epsilon\|Sw\|^2 + \epsilon\|dw\|^2,
\end{aligned}
$$

since $Sd\phi = 0$, $Sd\chi = 0$, and $w \in (\mathfrak{Z}^1)^{\perp_V}$ again. Thus,

$$
\begin{aligned}
a(\sigma, \phi, u; \tau, \psi, v) &\geq \frac{1}{2}\|\sigma\|^2 - \frac{\epsilon^2}{2}\|\rho\|^2 + \epsilon\|d\chi\|^2 + \|d\phi\|^2 + \epsilon\|Sw\|^2 + \epsilon\|dw\|^2 \\
&\geq -\frac{\epsilon^2}{2}\|\rho\|^2 + \epsilon\|d\chi\|^2 + \|d\phi\|^2 + \epsilon\left(\|Sw\|^2 + \|dw\|^2\right) \\
&\geq -\frac{\epsilon^2}{2}\|\rho\|^2 + \frac{\epsilon}{2C_p^2}\left(\|\rho\|^2 + \|\chi\|_V^2\right) + \frac{1}{2C_p^2}\left(\|\sigma\|^2 + \|\phi\|_V^2\right) + \frac{\epsilon}{C_p^2}\|w\|_V^2,
\end{aligned}
$$

by the Poincaré inequalities $\|\rho\|^2 + \|d\chi\|^2 \leq C_p^2\|d\chi\|^2$, $\|\sigma\|^2 + \|d\phi\|^2 \leq C_p^2\|d\phi\|^2$, and $\|w\|_V^2 \leq C_p^2\left(\|Sw\|^2 + \|dw\|^2\right)$, given by Proposition 5.1, and the Poincaré inequalities for V^0 and $\widetilde{V}^0$, $\|\chi\|_V \leq C_p\|d\chi\|$ and $\|\phi\|_V \leq C_p\|d\phi\|$. Thus, for small $\epsilon > 0$,

$$
a(\sigma, \phi, u; \tau, \psi, v) \geq C\left(\|\sigma\|_V^2 + \|\rho\|_V^2 + \|\phi\|_V^2 + \|\chi\|_V^2 + \|w\|_V^2\right) \geq C\|\tau, \psi, v\|_V\|\sigma, \phi, u\|_V,
$$

since $\|u\|_V^2 = \|d\chi\|^2 + \|w\|_V^2$. This concludes the proof. $\square$

5.2 Discretization

We just developed an abstract framework and obtained the system 5.3, the Hodge Laplacian of the complex (5.1). We now discretize this problem, identify a method to solve numerically this problem, and find a priori error estimates. We do so by going back to how the new complex was generated to find finite elements. We suppose that we have finite element spaces V_h^0, V_h^1, W_h^2, $\widetilde{V}_h^0$, $\widetilde{V}_h^1$, and $\widetilde{W}_h^2$, satisfying the *subcomplex property*, namely that they are subsets of their respective spaces and each form a complex. Moreover, we want them to be related in the following way,

$$\begin{tikzcd}
0 \arrow[r, hook] & V_h^0 \arrow[r, "d"] & V_h^1 \arrow[r, "d"] & V_h^2 \\
0 \arrow[r, hook] & \widetilde{V}_h^0 \arrow[r, "d"] \arrow[ur, "S_h"] & \widetilde{V}_h^1 \arrow[r, "d"] \arrow[ur, "S_h"] & \widetilde{V}_h^2
\end{tikzcd}$$

and equipped with canonical cochain projections Π_h^0 onto V_h^0, Π_h^1 onto V_h^1, Π_h^2 onto V_h^2, $\widetilde{\Pi}_h^0$ onto $\widetilde{V}_h^0$, $\widetilde{\Pi}_h^1$ onto $\widetilde{V}_h^1$, and $\widetilde{\Pi}_h^2$ onto $\widetilde{V}_h^2$. We also set $S_{0,h} := \Pi_h^1 S_0$ and $S_{1,h} := \Pi_h^2 S_1$, and see that $dS_{0,h} = -S_{1,h}d$. We suppose that $S_{0,h}$ is bounded uniformly in h from $\widetilde{V}_h^0$ to V_h^1. We also suppose that $S_{0,h}$ satisfies a *surjectivity hypothesis*, namely that

$$S_{0,h}\widetilde{\Pi}_h^0\psi = \Pi_h^1 S_0\psi, \tag{5.4}$$

for any ψ in the domain of $\widetilde{\Pi}_h^0$ such that $S\psi$ is in the domain of Π_h^1. Since $\Pi_h^1 S_0$ is surjective, this hypothesis implies that $S_{0,h}\widetilde{\Pi}_h^0$ also is, so that $S_{0,h}$ maps $\widetilde{V}_h^0$ onto V_h^1. We then have the following discrete version of the second order complex (5.1) with weak symmetries,

$$0 \to \Gamma_h \xrightarrow{(0\ d)} \widetilde{V}_h^1 \xrightarrow{\begin{pmatrix} S_{1,h} \\ d \end{pmatrix}} V_h^2 \times \widetilde{V}_h^2, \tag{5.5}$$

where

$$\Gamma_h = \left\{ (\sigma_h, \phi_h) \in V_h^0 \times \widetilde{V}_h^0 \mid d\sigma_h = S_{0,h}\phi_h \right\}.$$

Proposition 5.3. *Suppose the hypothesis of Proposition 5.1 holds. Suppose also we have finite element spaces forming two exact complexes, namely V_h^i and $\widetilde{V}_h^i$, for $0 \le i \le 2$, equipped with canonical cochain projections π_h^i and $\widetilde{\pi}_h^i$. We set $S_{0,h} := \Pi_h^1 S_0$ and*

$S_{1,h} := \Pi_h^2 S_1$, and suppose that $S_{0,h}$ is bounded uniformly in h from $\widetilde{V}_h^0$ to V_h^1, and satisfies the surjectivity hypothesis. The following discrete Poincaré inequalities hold with constants independent of h. For any $(\sigma_h, \phi_h) \in \Gamma_h$,

$$\|\sigma_h\| \le C\|d\phi_h\|,$$

For any $u_h \in \widetilde{V}_h^1$ such that $u_h \perp \mathcal{N}(d) \cap \mathcal{N}(S_h) \cap \widetilde{V}_h^1$,

$$\|u_h\| \le C\left(\|S_{1,h}u_h\| + \|du_h\|\right).$$

Proof. Take $(\sigma_h, \phi_h) \in \Gamma_h$. Then,

$$\|\sigma_h\| \le C_p\|d\sigma_h\| = C_p\|S_{0,h}\phi_h\| \le C\|\phi_h\| \le C\|d\phi_h\|,$$

using the Poincaré inequalities for V^0 and $\widetilde{V}^0$, and the hypothesis that $S_{0,h}$ is uniformly bounded in h. This concludes the first inequality.

We now turn to the second inequality. We show that, for any $(\eta_h, \rho_h) \in V_h^2 \times \widetilde{V}_h^2$ in the range of $\left(\begin{smallmatrix} S_h \\ d \end{smallmatrix}\right)$, there exists $u_h \in \widetilde{V}_h^1$ such that $\eta_h = S_h u_h$ and $\rho_h = du_h$ and $\|u_h\| \le C\left(\|\eta_h\| + \|\rho_h\|\right)$, with a constant independent of h.

We start by taking $v_h \in \widetilde{V}_h^1$ such that $\eta_h = S_h v_h$ and $\rho_h = dv_h$. We then use the continuous bottom complex to apply a Hodge decomposition, $v_h = d\psi + w$, for $\psi \in \widetilde{V}^0$ and $w \in (\widetilde{\mathfrak{Z}}^1)^\perp$, such that

$$\|w\|_V \le C\|dw\| = C\|dv\| = C\|\rho_h\|.$$

We then see that

$$-d\Pi_h S\widetilde{\Pi}_h\psi = \Pi_h S\widetilde{\Pi}_h d\psi = S_h v_h - S_h\widetilde{\Pi}_h w = \eta_h - S_h\widetilde{\Pi}_h w =: \eta_h',$$

so that η_h' is in the range of d. Thus, we can use the regular decomposition, which says that $V^1 = S\widetilde{V}^0 + dV^0$ continuously, to see that there exists $\phi \in \widetilde{V}^0$ with $-dS\phi = \eta_h'$ and

$$\|\phi\|_V \le C\|\eta_h'\| \le C\left(\|\eta_h\| + \|S_h\widetilde{\Pi}_h w\|\right) \le C\left(\|\eta_h\| + \|w\|\right) \le C\left(\|\eta_h\| + \|\rho_h\|\right).$$

We now set

$$u_h := \widetilde{\Pi}_h(w + d\phi) = \widetilde{\Pi}_h w + d\widetilde{\Pi}_h\phi,$$

and verify that $S_h u_h = \eta_h$ and $du_h = \rho_h$. We thus compute

$$S_h u_h = S_h(\widetilde{\Pi}_h w + d\widetilde{\Pi}_h \phi) = S_h\widetilde{\Pi}_h w - dS_h\widetilde{\Pi}_h\phi = S_h\widetilde{\Pi}_h w - d\Pi_h S\phi = S_h\widetilde{\Pi}_h w - \Pi_h dS\phi$$
$$= S_h\widetilde{\Pi}_h w + \Pi_h(\eta_h - S_h\widetilde{\Pi}_h w) = S_h\widetilde{\Pi}_h w + \Pi_h\eta_h - \Pi_h S_h\widetilde{\Pi}_h w = S_h\widetilde{\Pi}_h w + \eta_h - S_h\widetilde{\Pi}_h w = \eta_h,$$

and

$$du_h = d(\widetilde{\Pi}_h w + d\widetilde{\Pi}_h\phi) = \widetilde{\Pi}_h dw = \widetilde{\Pi}_h\rho_h = \rho_h,$$

as desired. Finally, we have that

$$\|u_h\| = \|d\phi\| + \|w\| \leq C\left(\|\eta_h\| + \|\rho_h\|\right),$$

with a constant independent of h, as desired. This concludes the proof. □

The method to solve (5.3) is then to find $(\sigma_h, \phi_h) \in V_h^0 \times \widetilde{V}_h^0$, $u_h \in \widetilde{V}_h^1$, and $\lambda_h \in V_h^1$ such that

$$(\sigma_h, \tau) - (u_h, d\psi) - (\lambda_h, d\tau - S_h\psi) = 0, \qquad (\tau, \psi) \in V_h^0 \times \widetilde{V}_h^0, \tag{5.6a}$$
$$(d\phi_h, v) + (S_h u_h, S_h v) + (du_h, dv) = (f, v), \qquad v \in \widetilde{V}_h^1, \tag{5.6b}$$
$$(d\sigma_h - S_h\phi_h, \mu) = 0, \qquad \mu \in V_h^1. \tag{5.6c}$$

We note that $d\sigma_h - S_h\phi_h = 0$ exactly. Since the Lagrange multiplier makes the system easier to discretized, this is the system that we discretize. We show this system is invertible.

Proposition 5.4. *Suppose the hypotheses of Proposition 5.3 hold. The discrete system* (5.6) *is invertible.*

Proof. The system is square, so we only need to show that given $f = 0$, the only solution is $(\sigma_h, \phi_h, u_h, \lambda_h) = (0, 0, 0, 0)$.

Taking $(\tau, \psi, v, \mu) = (\sigma_h, \phi_h, u_h, \lambda_h)$ in the equations (5.6) of the method, we get that $0 = \|\sigma_h\|^2 + \|S_h u_h\|^2 + \|du_h\|^2$, so that $\sigma_h = S_h u_h = du_h = 0$. In this case, the second equation with $v = d\phi_h$ says that $d\phi_h = 0$, and, since the complex is exact at $\widetilde{V}_h^0$, we have that $\phi_h = 0$.

We now need to show that $u_h = 0$. Using the Hodge decomposition of the discrete version of the complex (5.1), we first note that $u_h = d\chi_h + w_h$ for some $w_h \in (\mathfrak{Z}_h^1)^{\perp_{V_h}}$, and

$(\rho_h, \chi_h) \in V_h^0 \times \widetilde{V}_h^0$ such that $d\rho_h = S_h\chi_h$. Since $dw_h = du_h = 0$ and $S_h w_h = S_h u_h = 0$, we have $w_h \in \mathfrak{Z}_h^1$, so $w_h = 0$. Using the first equation of the discrete system with $\psi = \chi_h$ and $\tau = \rho_h$, we have that $-\|d\chi_h\|^2 = 0$, so that $\chi_h = 0$ using the Poincaré inequality $\|\chi_h\| \leq C_{p,h}\|d\chi_h\|$ for $\widetilde{V}_h^0$. Thus, $u_h = d\chi_h + w_h = 0$.

Finally, we need to show that $\lambda_h = 0$. To do so, we use the first equation of the discrete system with $\tau = 0$. Since $S_{0,h}$ is surjective, we set ψ to be a pre-image of λ_h under $S_{0,h}$, and see that $\lambda_h = 0$ since $u_h = 0$. This completes the proof. $\square$

We now define $\|w\|_{\Gamma_h}$ to be the graph norm of Γ_h, and show the following error estimate.

Theorem 5.5. *Suppose the hypotheses of Proposition 5.3 hold. Assuming also that $\|(I-\Pi_h)S\Psi\| \leq Ch\|\Psi\|_V$ for any $\Psi \in \widetilde{V}_h^0$, and $\|(I-\Pi_h)Sv\| \leq Ch\|v\|_V$ for any $v \in \widetilde{V}_h^1$, we have*

$$\begin{aligned}
&\|\sigma - \sigma_h\|_V^2 + \|\phi - \phi_h\|_V^2 + \|u - u_h\|_{\Gamma_h}^2 \\
&\leq C\left(\|\Pi_h\sigma - \sigma\|_V^2 + \|\widetilde{\Pi}_h u - u\|_{\Gamma_h}^2 + h^2\|\lambda\|^2 + \|\widetilde{\Pi}_h\phi - \phi\|_V^2 + h^2\|Su\|^2 + \|(I-\Pi_h)Su\|^2\right).
\end{aligned}$$

We start by writing the error equation.

Proposition 5.6. *Suppose the hypotheses of Proposition 5.3 hold. The error equation is*

$$\begin{aligned}
&(\Pi_h\sigma - \sigma_h, \tau) - (\widetilde{\Pi}_h u - u_h, d\psi) - (\lambda - \lambda_h, d\tau - S_h\psi) \\
&\qquad + (d(\widetilde{\Pi}_h\phi - \phi_h), v) + (S_h(\Pi_h u - u_h), S_h v) + (d(\widetilde{\Pi}_h u - u_h), dv) \\
&= (\Pi_h\sigma - \sigma, \tau) - (\widetilde{\Pi}_h u - u, d\psi) - (\lambda, (S - S_h)\psi) \\
&\qquad + (d(\widetilde{\Pi}_h\phi - \phi), v) - (Su, (S - S_h)v) - ((S - S_h)u, S_h v) + (S_h(\Pi_h u - u), S_h v) \\
&\qquad + (d(\widetilde{\Pi}_h u - u), dv).
\end{aligned}$$

Proof. We consider $\tau \in V_h^0$, $\psi \in \widetilde{V}_h^0$, $v \in \widetilde{V}_h^1$, and obtain the error equation. Thus, we take the difference between the equations for the exact solution and for the approximate solution,

$$\begin{aligned}
&(\sigma - \sigma_h, \tau) - (u - u_h, d\psi) - (\lambda, d\tau - S\psi) + (\lambda_h, d\tau - S_h\psi) \\
&\qquad\qquad + (d(\phi - \phi_h), v) + (Su, Sv) - (S_h u_h, S_h v) + (d(u - u_h), dv) = 0,
\end{aligned}$$

or, introducing projections,

$$
\begin{aligned}
(\Pi_h\sigma-\sigma_h,\tau)-(\widetilde{\Pi}_h u-u_h,d\psi)-(\lambda,d\tau-S\psi)+(\lambda_h,d\tau-S_h\psi)\\
+(d(\widetilde{\Pi}_h\phi-\phi_h),v)+(Su,Sv)-(S_hu_h,S_hv)+(d(\widetilde{\Pi}_h u-u_h),dv)\\
=(\Pi_h\sigma-\sigma,\tau)-(\widetilde{\Pi}_h u-u,d\psi)+(d(\widetilde{\Pi}_h\phi-\phi),v)+(d(\widetilde{\Pi}_h u-u),dv).
\end{aligned}
$$

Moreover, since

$$
-(\lambda,d\tau-S\psi)+(\lambda_h,d\tau-S_h\psi)=-(\lambda-\lambda_h,d\tau-S_h\psi)+(\lambda,(S-S_h)\psi),
$$

and

$$
\begin{aligned}
(Su,Sv)-(S_hu_h,S_hv)&=(Su,(S-S_h)v)+((S-S_h)u,S_hv)+(S_h(u-u_h),S_hv)\\
&=(Su,(S-S_h)v)+((S-S_h)u,S_hv)+(S_h(\Pi_h u-u_h),S_hv)\\
&\quad-(S_h(\Pi_h u-u),S_hv),
\end{aligned}
$$

we also have

$$
\begin{aligned}
&(\Pi_h\sigma-\sigma_h,\tau)-(\widetilde{\Pi}_h u-u_h,d\psi)-(\lambda-\lambda_h,d\tau-S_h\psi)\\
&\qquad+(d(\widetilde{\Pi}_h\phi-\phi_h),v)+(S_h(\Pi_h u-u_h),S_hv)+(d(\widetilde{\Pi}_h u-u_h),dv)\\
&=(\Pi_h\sigma-\sigma,\tau)-(\widetilde{\Pi}_h u-u,d\psi)-(\lambda,(S-S_h)\psi)\\
&\qquad+(d(\widetilde{\Pi}_h\phi-\phi),v)-(Su,(S-S_h)v)-((S-S_h)u,S_hv)+(S_h(\Pi_h u-u),S_hv)\\
&\qquad+(d(\widetilde{\Pi}_h u-u),dv),
\end{aligned}
$$

as desired. □

We are now ready to prove the inequality given in Theorem 5.5.

Proof of Theorem 5.5. We consider the error equation, and use the discrete decomposition $\widetilde{\Pi}_h u-u_h=d\chi_h+\omega_h$, where $\chi_h\in\widetilde{V}_h^0$, such that $S_h\chi_h=d\rho_h$ for some $\rho_h\in V_h^0$, and $\omega_h\in\widetilde{V}_h^1$ such that $\omega_h\perp\mathcal{N}\begin{pmatrix}S_h\\ d\end{pmatrix}\cap\widetilde{V}_h^1$. Hence,

$$
\begin{aligned}
&(\Pi_h\sigma-\sigma_h,\tau)-(d\chi_h+\omega_h,d\psi)\\
&\qquad-(\lambda-\lambda_h,d\tau-S_h\psi)+(d(\widetilde{\Pi}_h\phi-\phi_h),v)+(S_h\omega_h,S_hv)+(d\omega_h,dv)\\
&\qquad=(\Pi_h\sigma-\sigma,\tau)-(\widetilde{\Pi}_h u-u,d\psi)-(\lambda,(S-S_h)\psi)+(d(\widetilde{\Pi}_h\phi-\phi),v)\\
&\qquad\quad-(Su,(S-S_h)v)-((S-S_h)u,S_hv)+(S_h(\Pi_h u-u),S_hv)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+(d(\widetilde{\Pi}_h u-u),dv),
\end{aligned}
$$

since $S_h d\chi_h = -dS_h\chi_h = -d(d\rho_h) = 0$.

Now, we take $\tau = \Pi_h\sigma - \sigma_h - \epsilon\rho_h \in V^0$, $\psi = \widetilde{\Pi}_h\phi - \phi_h - \epsilon\chi_h \in \widetilde{V}^0$, and $v = d(\widetilde{\Pi}_h\phi - \phi_h) + d\chi_h + \epsilon\omega_h \in \widetilde{V}^1$. Using the surjectivity hypothesis (5.4), the constraints, and the commutativity property of d and the projections, we see that

$$\begin{aligned} d\tau = d(\Pi_h\sigma - \sigma_h) - \epsilon d\rho_h = \Pi_h d\sigma - d\sigma_h - \epsilon S_h\chi_h = \Pi_h S\phi - S_h\phi_h - \epsilon\Pi_h S\chi_h \\ = \Pi_h S\widetilde{\Pi}_h\phi - \Pi_h S\phi_h - \epsilon\Pi_h S\chi_h = \Pi_h S(\widetilde{\Pi}_h\phi - \phi_h - \epsilon\chi_h) = S_h\psi, \end{aligned}$$

for this particular choice of test functions. Thus, $(\tau, \psi, v) \in Z_h = \Gamma_h \times \widetilde{V}_h^1$ for the choice of test functions made. Using this, the left hand side of the error equation becomes

$$\begin{aligned} &\|\Pi_h\sigma - \sigma_h\|^2 - \epsilon(\Pi_h\sigma - \sigma_h, \rho_h) - (d\chi_h + \omega_h, d(\widetilde{\Pi}_h\phi - \phi_h)) + \epsilon\|d\chi_h\|^2 + \epsilon(\omega_h, d\chi_h) \\ &+ \|d(\widetilde{\Pi}_h\phi - \phi_h)\|^2 + (d(\widetilde{\Pi}_h\phi - \phi_h), d\chi_h) + \epsilon(d(\widetilde{\Pi}_h\phi - \phi_h), \omega_h) + \epsilon\|S_h\omega_h\|^2 + \epsilon\|d\omega_h\|^2 \\ &\qquad \geq \|\Pi_h\sigma - \sigma_h\|^2 - \frac{1}{2}\|\Pi_h\sigma - \sigma_h\|^2 - \frac{\epsilon^2}{2}\|\rho_h\|^2 + \epsilon\|d\chi_h\|^2 + \|d(\widetilde{\Pi}_h\phi - \phi_h)\|^2 \\ &\qquad\qquad + \epsilon\|S_h\omega_h\|^2 + \epsilon\|d\omega_h\|^2, \end{aligned}$$

since $\omega_h \perp \mathcal{N}\begin{pmatrix} S_h \\ d \end{pmatrix} \cap \widetilde{V}_h^1$, and $S_h v = S_h\omega_h$. Using the Poincaré inequalities for V_h^0, $\widetilde{V}_h^0$, and Proposition 5.3,

$$\begin{aligned} &\frac{1}{2}\|\Pi_h\sigma - \sigma_h\|^2 - \frac{\epsilon^2}{2}\|\rho_h\|^2 + \epsilon\|d\chi_h\|^2 + \|d(\widetilde{\Pi}_h\phi - \phi_h)\|^2 + \epsilon\|S_h\omega_h\|^2 + \epsilon\|d\omega_h\|^2 \\ &\qquad \geq \frac{1}{2}\|\Pi_h\sigma - \sigma_h\|^2 - \frac{\epsilon^2}{2}\|\rho_h\|^2 + \frac{\epsilon}{C_p^2}\left(\|\rho_h\|^2 + \|\chi_h\|_V^2\right) \\ &\qquad\qquad + \frac{1}{C_p^2}\left(\|\Pi_h\sigma - \sigma_h\| + \|\widetilde{\Pi}_h\phi - \phi_h\|_V^2\right) + \frac{\epsilon}{C_p^2}\|\omega_h\|_{\Gamma_h}^2. \end{aligned}$$

Now, we can again use Proposition 5.3 to see that $\|\rho_h\| \leq C\|\chi_h\|$. Thus, for small $\epsilon > 0$,

$$\begin{aligned} &\frac{1}{2}\|\Pi_h\sigma - \sigma_h\|^2 - \frac{\epsilon^2}{2}\|\rho_h\|^2 \\ &\quad + \frac{\epsilon}{C_p^2}\left(\|\rho_h\|^2 + \|\chi_h\|_V^2\right) + \frac{1}{C_p^2}\left(\|\Pi_h\sigma - \sigma_h\| + \|\widetilde{\Pi}_h\phi - \phi_h\|_V^2\right) + \frac{\epsilon}{C_p^2}\|\omega_h\|_{\Gamma_h}^2 \\ &\qquad \geq \frac{1}{C}\left(\|\Pi_h\sigma - \sigma_h\|_V^2 + \|\widetilde{\Pi}_h\phi - \phi_h\|_V^2 + \|\rho_h\|_V^2 + \|\chi_h\|_V^2 + \|\omega_h\|_{\Gamma_h}^2\right). \end{aligned}$$

Returning to the error equation, we then have

$$\frac{1}{C}\left(\|\Pi_h\sigma-\sigma_h\|_V^2+\|\widetilde{\Pi}_h\phi-\phi_h\|_V^2+\|\rho_h\|_V^2+\|\chi_h\|_V^2+\|\omega_h\|_{\Gamma_h}^2\right)$$
$$\le \|\Pi_h\sigma-\sigma\|\|\Pi_h\sigma-\sigma_h\|+\epsilon\|\Pi_h\sigma-\sigma\|\|\rho_h\|+\|\widetilde{\Pi}_h u-u\|\left(\|d(\widetilde{\Pi}_h\phi-\phi_h)\|+\epsilon\|d\chi_h\|\right)$$
$$+\|\lambda\|\|(I-\Pi_h)S(\widetilde{\Pi}_h\phi-\phi_h-\epsilon\chi_h)\|+\|d(\widetilde{\Pi}_h\phi-\phi)\|\left(\|d(\widetilde{\Pi}_h\phi-\phi_h)\|+\|d\chi_h\|+\epsilon\|\omega_h\|\right)$$
$$+\|Su\|\|(I-\Pi_h)S(d(\widetilde{\Pi}_h\phi-\phi_h)+d\chi_h+\epsilon\omega_h)\|+\|(I-\Pi_h)Su\|\|S_h\omega_h\|$$
$$+\|S_h(\Pi_h u-u)\|\|S_h\omega_h\|+\|d(\widetilde{\Pi}_h u-u)\|\|d\omega_h\|,$$

so that

$$\begin{aligned}&\|\Pi_h\sigma-\sigma_h\|_V^2+\|\widetilde{\Pi}_h\phi-\phi_h\|_V^2+\|\rho_h\|_V^2+\|\chi_h\|_V^2+\|\omega_h\|_{\Gamma_h}^2\\ &\le C\Big(\|\Pi_h\sigma-\sigma\|^2+\|\widetilde{\Pi}_h u-u\|^2+\|\lambda\|\|(I-\Pi_h)S(\widetilde{\Pi}_h\phi-\phi_h-\epsilon\chi_h)\|\\ &\quad+\|d(\widetilde{\Pi}_h\phi-\phi)\|^2+\|Su\|\|(I-\Pi_h)S(d(\widetilde{\Pi}_h\phi-\phi_h)+d\chi_h+\epsilon\omega_h)\|\\ &\quad+\|(I-\Pi_h)Su\|^2+\|S_h(\Pi_h u-u)\|^2+\|d(\widetilde{\Pi}_h u-u)\|^2\Big).\end{aligned}$$

Assuming that $\|(I-\Pi_h)S\Psi\|\le Ch\|\Psi\|_V$ for any $\Psi\in\widetilde{V}_h^0$, and $\|(I-\Pi_h)Sv\|\le Ch\|v\|_V$ for any $v\in\widetilde{V}_h^1$, we get that

$$\begin{aligned}&\|\Pi_h\sigma-\sigma_h\|_V^2+\|\widetilde{\Pi}_h\phi-\phi_h\|_V^2+\|\rho_h\|_V^2+\|\chi_h\|_V^2+\|\omega_h\|_{\Gamma_h}^2\\ &\le C\Big(\|\Pi_h\sigma-\sigma\|^2+\|\widetilde{\Pi}_h u-u\|^2+h^2\|\lambda\|^2+\|d(\widetilde{\Pi}_h\phi-\phi)\|^2+h^2\|Su\|^2\\ &\quad+\|(I-\Pi_h)Su\|^2+\|S_h(\Pi_h u-u)\|^2+\|d(\widetilde{\Pi}_h u-u)\|^2\Big).\end{aligned}$$

We then write the left hand side in terms of $\widetilde{\Pi}_h u-u_h$,

$$\begin{aligned}&\|\Pi_h\sigma-\sigma_h\|_V^2+\|\widetilde{\Pi}_h\phi-\phi_h\|_V^2+\|\widetilde{\Pi}_h u-u_h\|_{\Gamma_h}^2\\ &\le C\Big(\|\Pi_h\sigma-\sigma\|^2+\|\widetilde{\Pi}_h u-u\|^2+h^2\|\lambda\|^2+\|d(\widetilde{\Pi}_h\phi-\phi)\|^2+h^2\|Su\|^2\\ &\quad+\|(I-\Pi_h)Su\|^2+\|S_h(\Pi_h u-u)\|^2+\|d(\widetilde{\Pi}_h u-u)\|^2\Big).\end{aligned}$$

Finally, applying the triangle inequality, we get the desired inequality. □

5.3 Application to the Time-Independent EB System

We now apply the abstract framework and its discretization to the complex (4.16), as mentioned in the introduction of this chapter. In this application, we have

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathring{H}^1 \arrow[r, "\operatorname{grad}"] & \mathring{H}(\operatorname{curl}) \arrow[r, "\operatorname{curl}"] & L^2(\mathbb{V}) \\
0 \arrow[r, hook] & \mathring{H}^1(\mathbb{V}) \arrow[r, "\operatorname{grad}"] \arrow[ur, "I"] & \mathring{H}(\operatorname{curl}, \mathbb{M}) \arrow[r, "\operatorname{curl}"] \arrow[ur, "\operatorname{vskw}"] & L^2(\mathbb{M})
\end{tikzcd}$$

along with $W^0 = L^2(\mathbb{R})$, $W^1 = L^2(\mathbb{V})$, $W^2 = L^2(\mathbb{V})$, $\widetilde{W}^0 = L^2(\mathbb{V})$, $\widetilde{W}^1 = L^2(\mathbb{M})$, and $\widetilde{W}^2 = L^2(\mathbb{M})$. The spaces are summarized in Table 5.1. We have that the regularity property $\operatorname{curl} \mathring{H}(\operatorname{curl}) = \operatorname{curl} \mathring{H}^1(\mathbb{V})$ is satisfied, from the regular decomposition $\mathring{H}(\operatorname{curl}) = \mathring{H}^1(\mathbb{V}) + \operatorname{grad} H^1$, [27, Section 3] and [28, Lemma 2.4]. The problem given by equations (5.3) is to find $(\sigma, \phi) \in \mathring{H}^1(\mathbb{R}) \times \mathring{H}^1(\mathbb{V})$, $u \in \mathring{H}(\operatorname{curl}, \mathbb{M})$, and $\lambda \in H^{-1}(\operatorname{div}, \mathbb{V})$ such that

$$\begin{aligned}
(\sigma, \tau) - (u, \operatorname{grad} \psi) - (\lambda, \operatorname{grad} \tau - I\psi) &= 0, & (\tau, \psi) &\in \mathring{H}^1 \times \mathring{H}^1(\mathbb{V}), \\
(\operatorname{grad} \phi, v) + (\operatorname{skw} u, \operatorname{skw} v) + (\operatorname{curl} u, \operatorname{curl} v) &= (f, v), & v &\in \mathring{H}(\operatorname{curl}, \mathbb{M}), \\
(\operatorname{grad} \sigma - I\phi, \mu) &= 0, & \mu &\in H^{-1}(\operatorname{div}, \mathbb{V}),
\end{aligned}$$

where

$$H^{-1}(\operatorname{div}, \mathbb{V}) := \left\{ \lambda \in H^{-1}(\mathbb{V}) \mid \operatorname{div} \lambda \in H^{-1}(\mathbb{R}) \right\},$$

is the completion $\overline{W}^1$ of $W^1 = L^2(\mathbb{V})$ in the norm

$$|||\lambda||| := \sup_{(\tau,\psi) \in \mathring{H}^1(\mathbb{R}) \times \mathring{H}^1(\mathbb{V})} \frac{(\lambda, \operatorname{grad} \tau - I\psi)}{\|\tau\|_{H^1} + \|\psi\|_{H^1}}.$$

In order to find finite elements for this problem, we recall that they need to satisfy

- the subcomplex property,
- the existence of bounded cochain projections,
- the surjectivity hypothesis.

We choose the following finite element spaces: $V_h^0 = \mathcal{P}_1 \Lambda^0$, the Lagrange $\mathcal{P}_1$ elements, $V_h^1 = \mathcal{P}_1^- \Lambda^1$, the lowest order Nédélec $H(\operatorname{curl})$ elements of the first kind, $V_h^2 = \mathcal{P}_1^- \Lambda^2$,

V	W	V_h	Variables
$\mathring{H}(\mathbb{R})$	$L^2(\mathbb{R})$	$\mathcal{P}_1\Lambda^0$	σ, τ
$\mathring{H}(\mathbb{V})$	$L^2(\mathbb{V})$	$\mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$	ϕ, ψ
$\mathring{H}(\mathrm{curl}, \mathbb{V})$	$H^{-1}(\mathrm{div}, \mathbb{V})$	$\mathcal{P}_1^-\Lambda^1$	λ, μ
$\mathring{H}(\mathrm{curl}, \mathbb{M})$	$L^2(\mathbb{M})$	$\mathcal{P}_2^-\Lambda^1 \otimes \mathbb{V}$	u, v
$L^2(\mathbb{V})$	$L^2(\mathbb{V})$	$\mathcal{P}_1^-\Lambda^2$	
$L^2(\mathbb{M})$	$L^2(\mathbb{M})$	$\mathcal{P}_1\Lambda^2 \otimes \mathbb{V}$	

Table 5.1: Spaces used in the formulation.

the lowest order Raviart-Thomas elements, $\widetilde{V}_h^0 = \mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$, the vector-valued Lagrange $\mathcal{P}_2$ elements, $\widetilde{V}_h^1 = \mathcal{P}_2^-\Lambda^1 \otimes \mathbb{V}$, and $\widetilde{V}_h^2 = \mathcal{P}_1\Lambda^2 \otimes \mathbb{V}$. Thus, we have

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathcal{P}_1\Lambda^0 \arrow[r, "\mathrm{grad}"] & \mathcal{P}_1^-\Lambda^1 \arrow[r, "\mathrm{curl}"] & \mathcal{P}_1^-\Lambda^2 \\
0 \arrow[r, hook] & \mathcal{P}_2\Lambda^0 \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\Pi_h^1"] & \mathcal{P}_2^-\Lambda^1 \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\Pi_h^2\,\mathrm{vskw}"] & \mathcal{P}_1\Lambda^2 \otimes \mathbb{V}
\end{tikzcd}$$

and impose Dirichlet boundary conditions for the 0-forms, null tangent component for the 1-forms, and null normal component for the 2-forms. The finite elements used are summarized in Table 5.1. We use the canonical projections, which satisfy the surjectivity hypothesis (5.4). The problem is then to find $\sigma_h \in \mathcal{P}_1\Lambda^0$, $\phi_h \in \mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$, $u \in \mathcal{P}_2^-\Lambda^1 \otimes \mathbb{V}$, and $\lambda_h \in \mathcal{P}_1^-\Lambda^1$ such that

$$(\sigma_h, \tau) - (u_h, \mathrm{grad}\,\psi) - (\lambda_h, \mathrm{grad}\,\tau - \Pi_h^1\psi) = 0, \tag{5.7a}$$

$$(\mathrm{grad}\,\phi_h, v) + (\Pi_h^2\,\mathrm{vskw}\,u_h, \Pi_h^2\,\mathrm{vskw}\,v) + (\mathrm{curl}\,u_h, \mathrm{curl}\,v) = (f, v), \tag{5.7b}$$

$$(\mathrm{grad}\,\sigma_h - \Pi_h^1\phi_h, \mu) = 0, \tag{5.7c}$$

for any $\tau \in \mathcal{P}_1\Lambda^0$, $\psi \in \mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$, $v \in \mathcal{P}_2^-\Lambda^1 \otimes \mathbb{V}$, $\mu \in \mathcal{P}_1^-\Lambda^1$. We note the use of the canonical projections Π_h^1 from Lagrange to Nédélec $H(\mathrm{curl})$ elements, and Π_h^2 from the skew-part of Nédélec $H(\mathrm{curl})$ elements viewed as vectors to Raviart-Thomas elements. They appear due to the presence of S_0 and S_1 at the continuous level.

Proposition 5.7. *With this choice of finite elements, the surjectivity hypothesis* (5.4) *holds.*

Proof. By density, it is enough to consider $\psi \in \widetilde{\Lambda}^0$. We need to show that

$$\left(\Pi_h^1 S_0 - S_{0,h}\widetilde{\Pi}_h^0\right)\psi = 0.$$

If we set $\omega = (I - \widetilde{\Pi}_h^0)\psi \in \widetilde{\Lambda}^0$, then the condition becomes $\Pi_h^1 S_0 \omega = 0$, so that we need to show that all the degrees of freedom of ω are zero,

$$\int_f \operatorname{tr}_f(S_0\omega) \wedge \mu = 0$$

for any $\mu \in \mathcal{P}_{1+1-d-1}\Lambda^{d-1}(f)$, $f \in \Delta_d(\mathcal{T}_h)$, $1 \le d \le 1$. More precisely, we need to show that

$$\int_f S_0\omega \wedge \mu = 0$$

for any $\mu \in \mathcal{P}_0\Lambda^0(f)$, $f \in \Delta_1(\mathcal{T}_h)$. These degrees of freedom can further be written as

$$\int_f \omega \wedge \zeta = 0 \tag{5.8}$$

where $\zeta = (S_0)'\mu \in \mathcal{P}_0\widetilde{\Lambda}^1(f)$. (Indeed, using Leibniz rule, we have that $(S\omega) \wedge \mu = (-1)^k \omega \wedge S'\mu$, for $\omega \in \widetilde{\Lambda}^k$, where $S' = dK' - K'd : \Lambda^k \to \widetilde{\Lambda}^{k+1}$ and K' is the adjoint of K.) However, the last equation holds. Indeed, we have that $\widetilde{\Pi}_h^0\omega = 0$, so that the degrees of freedom are zero,

$$\int_f \operatorname{tr}_f \omega \wedge \zeta = 0$$

for any $\zeta \in \mathcal{P}_{2-d}^-\widetilde{\Lambda}^d(f)$, $f \in \Delta_d(\mathcal{T}_h)$, and $0 \le d \le 1$. In particular, for $d = 1$,

$$\int_f \omega \wedge \zeta = 0$$

for any $\zeta \in \mathcal{P}_1^-\widetilde{\Lambda}^1(f)$, $f \in \Delta_1(\mathcal{T}_h)$, so that equation (5.8) certainly holds. This concludes the proof. □

Proposition 5.8. *Consider the canonical projection $\widetilde{\Pi}_h^0$ mapping onto $\mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$. We have that $S_{0,h} = \widetilde{\Pi}_h^0$ is bounded uniformly in h from $\mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$ to $\mathcal{P}_1^-\Lambda^1$,* *Proof.* We note that $\|\psi\|_h := h^{3/2}\sum_{v\in\Delta_0}|\psi(v)| + h^{1/2}\sum_{e\in\Delta_1}|\int_e \psi|$ is a norm for $\psi \in \mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$, and $\|u\|_h := h^{1/2}\sum_{e\in\Delta_1}|\int_e u\cdot\tau_e|$ for $u \in \mathcal{P}_1^-\Lambda^1$. These two norms are equivalent to $\|\psi\|$ and $\|u\|$, respectively. Then, for $\psi \in \mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$, we see that

$$\begin{aligned}\|S_{0,h}\psi\| \le C\|S_{0,h}\psi\|_h \le C\|\Pi_h I\psi\|_h \le C\left(\sum_{e\in\Delta_1} h^{1/2}\left|\int_e (\Pi_h I\psi)\cdot\tau_e\right|\right)\\ = C\left(\sum_{e\in\Delta_1} h^{1/2}\left|\int_e \psi\cdot\tau_e\right|\right)\\ \le C\left(\sum_{e\in\Delta_1} h^{1/2}\left|\int_e \psi\right|\right) \le C\left(h^{3/2}\sum_{v\in\Delta_0}|\psi(v)| + h^{1/2}\sum_{e\in\Delta_1}\left|\int_e\psi\right|\right)\\ = C\|\psi\|_h \le C\|\psi\|,\end{aligned}$$

with a constant independent of h, as desired.

We note that $\|v\|_h := h^{-1/2}\sum_{e\in\Delta_1}\sum_{p\in\mathfrak{B}}|\int_e (v\times n)p| + h^{1/2}\sum_{F\in\Delta_2}|v(F)|$, where $\mathfrak{B}$ is a basis of $\mathcal{P}_1\Lambda^0$, is a norm for $v \in \mathcal{P}_2^-\Lambda^0 \otimes \mathbb{V}$, and $\|u\|_h := h^{1/2}\sum_{e\in\Delta_1}|\int_e u\cdot\tau_e|$ for $u \in \mathcal{P}_1^-\Lambda^1$. These two norms are equivalent to $\|\psi\|$ and $\|u\|$, respectively. Then, for $\psi \in \mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$, we see that

$$\begin{aligned}\|S_{0,h}\psi\| \le C\|S_{0,h}\psi\|_h \le C\|\Pi_h I\psi\|_h \le C\left(\sum_{e\in\Delta_1} h^{1/2}\left|\int_e (\Pi_h I\psi)\cdot\tau_e\right|\right)\\ = C\left(\sum_{e\in\Delta_1} h^{1/2}\left|\int_e \psi\cdot\tau_e\right|\right) \le C\left(\sum_{e\in\Delta_1} h^{1/2}\left|\int_e \psi\right|\right)\\ \le C\left(h^{3/2}\sum_{v\in\Delta_0}|\psi(v)| + h^{1/2}\sum_{e\in\Delta_1}\left|\int_e\psi\right|\right) = C\|\psi\|_h \le C\|\psi\|,\end{aligned}$$

with a constant independent of h, as desired. □

Moreover, we have that the hypothesis of Theorem 5.5 is satisfied.

Proposition 5.9. *Consider the canonical projections Π_h^1 mapping onto $\mathcal{P}_1^-\Lambda^1$ and Π_h^2 onto $\mathcal{P}_1^-\Lambda^2$. We have that $\|(I-\Pi_h^1)\psi\| \le Ch\|\psi\|_{H^1}$ for any $\psi \in \mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$ and $\|(I-\Pi_h^2)\operatorname{vskw} v\| \le Ch\|v\|_{H(\mathrm{curl})}$ for any $v \in \mathcal{P}_1\Lambda^1 \otimes \mathbb{V}$ are satisfied.*

Proof. For any tetrahedron T, since, on a finite dimensional space, all linear maps are bounded and all norms are equivalent,

$$\begin{aligned}\|(I-\Pi_T^1)I\psi\|_T &= \inf_{\chi\in\mathcal{P}_0\Lambda^0(T)\otimes\mathbb{V}} \|(\psi-\chi)-\Pi_T^1(\psi-\chi)\|_T \\ &\le C_T \inf_{\chi\in\mathcal{P}_0\Lambda^0(T)\otimes\mathbb{V}} \|\psi-\chi\|_{H^1(T)} \le C_T|\psi-\chi|_{H^1(T)} = C_T|\psi|_{H^1(T)},\end{aligned}$$

for any $\psi \in \mathcal{P}_2\Lambda^0(T)$. Thus, we have

$$\|(I-\Pi_h^1)I\psi\| \le Ch|\psi|_{H^1},$$

for any $\psi \in \mathcal{P}_2\Lambda^0$, where the constant depends only on the shape regularity of the mesh. This concludes the first estimate.

For any tetrahedron T, since, on a finite dimensional space, all linear maps are bounded and all norms are equivalent,

$$\begin{aligned}\|(I-\Pi_T^2)\operatorname{vskw} v\|_T &= \inf_{w\in\mathcal{P}_0\Lambda^1(T)\otimes\mathbb{V}} \|\operatorname{vskw}(v-w)-\Pi_T^2\operatorname{vskw}(v-w)\|_T \\ \le C_T \inf_{w\in\mathcal{P}_0\Lambda^1(T)\otimes\mathbb{V}} \|\operatorname{vskw}(v-w)\|_{H(\mathrm{curl},T)} &\le C_T \inf_{w\in\mathcal{P}_0\Lambda^1(T)\otimes\mathbb{V}} \|v-w\|_{H(\mathrm{curl},T)} \\ &\le C_T|v-w|_{H(\mathrm{curl},T)} = C_T|v|_{H(\mathrm{curl},T)},\end{aligned}$$

for any $v \in \mathcal{P}_1\Lambda^1(T)$. Thus, we have

$$\|(I-\Pi_h^2)\operatorname{vskw} v\| \le Ch|v|_{H(\mathrm{curl})},$$

for any $v \in \mathcal{P}_1\Lambda^1$, where the constant depends only on the shape regularity of the mesh. This concludes the second estimate. $\square$

We have verified the hypotheses of the abstract framework for the complex (4.16) and its discretization, as mentioned in the introduction. Therefore, the theory developed in this chapter thus applies to this particular case. However, we will not show the resulting precise estimates, as we are more interested in the case of the wave equation.

Chapter 6

Time-Dependent BGG Construction

We now carry the abstract framework developed in the previous chapter for the Hodge Laplacian to the Hodge wave equation. This enables us to study the time-dependent problem introduced in Section 4.11 rather than the time-independent problem discussed in the previous chapter.

We first revisit the abstract framework of the previous chapter and associate a time-dependent problem. We then find a discretization and apply this to the linearized EB system.

6.1 Abstract Framework

We recall the framework developed in Chapter 5. We are thus given two exact Hilbert complexes linked by bounded linear maps $S_0 : \widetilde{W}^0 \to W^1$, and $S_1 : \widetilde{W}^1 \to W^2$ such that S_0 is injective, $S_0\widetilde{V}^0 \subset V^1$, and $d^1 S_0 = -S_1 d^0$ on $\widetilde{V}^0$,

$$\begin{tikzcd}
0 \arrow[r, hook] & V^0 \arrow[r, "d"] & V^1 \arrow[r, "d"] & W^2 \\
0 \arrow[r, hook] & \widetilde{V}^0 \arrow[r, "d"] \arrow[ur, "S"] & \widetilde{V}^1 \arrow[r, "d"] \arrow[ur, "S"] & \widetilde{W}^2
\end{tikzcd}$$

with Hilbert spaces W^0, W^1, W^2, $\widetilde{W}^0$, $\widetilde{W}^1$, and $\widetilde{W}^2$. A key assumption is that a regularity property holds: $dS\widetilde{V}^0 = dV^1$. This setup allows us to build the closed complex

(5.1). The associated well-posed Hodge wave problem is to find

$$\begin{aligned}(\sigma,\phi) &\in C^0([0,T],\Gamma)\cap C^1([0,T],\bar\Gamma),\\ u &\in C^0([0,T],\widetilde V^1)\cap C^1([0,T],\widetilde W^1),\\ (M,\rho) &\in C^1([0,T],W_2\times\widetilde W_2),\end{aligned}$$

such that

$$(\dot\sigma,\tau)-(u,d\psi)=0, \qquad (\tau,\psi)\in\Gamma, \tag{6.1a}$$
$$(\dot u,v)+(d\phi,v)+(M,Sv)+(\rho,dv)=(f,v), \qquad v\in\widetilde V^1, \tag{6.1b}$$
$$(\dot M,N)-(Su,N)=0, \qquad N\in V^2, \tag{6.1c}$$
$$(\dot\rho,\mu)-(du,\mu)=0, \qquad \mu\in\widetilde V^2, \tag{6.1d}$$

with $f\in\widetilde W^1$, and initial conditions in $(\sigma,\phi,u,M,\rho)(0)\in\Gamma\times\widetilde V^1\times(W_2\times\widetilde W_2)$.

To impose the constraint used to define Γ, we introduce a multiplier $\lambda:=(S^*)^{-1}d^*u\in W^1$ and denote by $\overline W^1$ the completion of W^1 with the norm $|||\lambda|||$ defined in Chapter 5. The modified problem is to find

$$\begin{aligned}(\sigma,\phi) &\in C^0([0,T],\Gamma)\cap C^1([0,T],\bar\Gamma),\\ u &\in C^0([0,T],\widetilde V^1_0)\cap C^1([0,T],\widetilde W^1),\\ \rho &\in C^0([0,T],\widetilde V_2)\cap C^1([0,T],\widetilde W_2),\\ \lambda &\in C^0([0,T],W_1)\cap C^1([0,T],\overline W_1),\end{aligned}$$

such that

$$(\dot\sigma,\tau)-(u,d\psi)-(\lambda,d\tau-S\psi)=0, \qquad (\tau,\psi)\in V^0\times\widetilde V^0, \tag{6.2a}$$
$$(\dot u,v)+(d\phi,v)+(M,Sv)+(\rho,dv)=(f,v), \qquad v\in\widetilde V^1, \tag{6.2b}$$
$$(\dot M,N)-(Su,N)=0, \qquad N\in V^2, \tag{6.2c}$$
$$(\dot\rho,\mu)-(du,\mu)=0, \qquad \mu\in\widetilde V^2, \tag{6.2d}$$
$$(d\sigma-S\phi,\zeta)=0, \qquad \zeta\in W^1, \tag{6.2e}$$

with $f\in\widetilde W^1$, and $(\sigma,\tau,u,M,\rho,\lambda)(0)\in\Gamma\times\widetilde V^1_0\times(\widetilde W_2\times W_2)\times W_1$. Uniqueness is shown using an energy argument. Existence follows from the following proposition.

Proposition 6.1. *The systems* (6.2) *and* (6.1) *are equivalent in the following sense.*

- *Suppose* $(\sigma_0, \phi_0) \in \Gamma$, $u_0 \in \widetilde{V}^1$, $M_0 \in W^2$, $\rho_0 \in \widetilde{W}^2$, *and* $\lambda_0 \in W^1$. *If* $(\sigma, \phi, u, M, \rho, \lambda)$ *is a solution of* (6.2) *with* $(\sigma, \phi, u, M, \rho, \lambda)(0) = (\sigma_0, \phi_0, u_0, M_0, \rho_0, \lambda_0)$, *then* $d\sigma = S\phi$, *and* $(\sigma, \phi, u, M, \rho)$ *is a solution to* (6.1) *with* $(\sigma, \phi, u, M, \rho)(0) = (\sigma_0, \phi_0, u_0, M_0, \rho_0)$.
- *Suppose* $(\sigma_0, \phi_0) \in \Gamma$, $u_0 \in \widetilde{V}_0^1 \cap (\widetilde{V}_0^1)^*$, *and* $\rho_0 \in \widetilde{V}_2^*$. *Suppose also that* $(\sigma, \phi, u, 0, \rho)$ *is a solution of* (6.1) *with* $(\sigma, \phi, u, M, \rho)(0) = (\sigma_0, \phi_0, u_0, 0, \rho_0)$. *Moreover, we pick any* $\lambda_0 \in W^1$. *Then, if we set* $\lambda := (S^*)^{-1} d^* u$, *we have that* $(\sigma, \phi, u, M, \rho, \lambda)$ *is a solution to* (6.1) *with* $(\sigma, \phi, u, M, \rho, \lambda)(0) = (\sigma_0, \phi_0, u_0, 0, \rho_0, \lambda_0)$.

Proof. Suppose $(\sigma_0, \phi_0) \in \Gamma$, $u_0 \in \widetilde{V}^1$, $M_0 \in W^2$, $\rho_0 \in \widetilde{W}^2$, and $\lambda_0 \in V^1$. Let $(\sigma, \phi, u, M, \rho, \lambda)$ be a solution of (6.2) with $(\sigma, \phi, u, M, \rho, \lambda)(0) = (\sigma_0, \phi_0, u_0, 0, \rho_0, \lambda_0)$. Since $d\sigma$ and $S\phi$ are in W^1, the last equation of (6.2) gives that $d\sigma = S\phi$. Restricting the first equation to Γ shows that $(\sigma, \phi, u, M, \rho, \lambda)$ also satisfies (6.1). This completes the first part of the proposition.

Suppose $(\sigma_0, \phi_0) \in \Gamma$, $u_0 \in \widetilde{V}_0^1 \cap (\widetilde{V}_0^1)^*$, $\rho_0 \in \widetilde{V}_2^*$, and $\lambda_0 \in W^1$. Let $(\sigma, \phi, u, 0, \rho)$ be a solution of (6.1) with $(\sigma, \phi, u, M, \rho)(0) = (\sigma_0, \phi_0, u_0, 0, \rho_0)$. By Theorem 4.7,

$$u \in C^0([0,T], \widetilde{V}_0^1 \cap (\widetilde{V}_0^1)^*) \cap C^1([0,T], W^1),$$

so we can let $\lambda := (S^*)^{-1} d^* u$. With this definition, for any $(\tau, \psi) \in V^0 \times \widetilde{V}^0$,

$$\begin{aligned}
(\dot{\sigma}, \tau) - (u, d\psi) - (\lambda, d\tau - S\psi) &= (\dot{\sigma}, \tau) - (u, d\psi) - ((S^*)^{-1} d^* u, d\tau - S\psi) \\
&= (\dot{\sigma}, \tau) - (u, d\psi) - ((S^{-1})^* d^* u, d\tau - S\psi) \\
&= (\dot{\sigma}, \tau) - (u, d\psi) - (u, dS^{-1} d\tau - dS^{-1} S\psi) \\
&= (\dot{\sigma}, \tau) - (u, d\psi) - (u, dS^{-1} d\tau - d\psi) = (\dot{\sigma}, \tau) - (u, dS^{-1} d\tau) = 0
\end{aligned}$$

since $(\tau, dS^{-1} d\tau) \in \Gamma$. Thus, $(\sigma, \phi, u, M, \rho, \lambda)$ is a solution to (6.1) with $(\sigma, \phi, u, M, \rho, \lambda)(0) = (\sigma_0, \phi_0, u_0, 0, \rho_0, \lambda_0)$. □

6.2 Discretization

We now discretize the abstract Hodge wave (6.2) associated to the complex (5.1). We assume that we have the finite element spaces V_h^0, V_h^1, V_h^2, $\widetilde{V}_h^0$, $\widetilde{V}_h^1$, and $\widetilde{V}_h^2$, forming subcomplexes of the original complexes. Moreover, we assume they are related as

$$\begin{tikzcd}
0 \arrow[r, hook] & V_h^0 \arrow[r, "d"] & V_h^1 \arrow[r, "d"] & V_h^2 \\
0 \arrow[r, hook] & \widetilde{V}_h^0 \arrow[r, "d"] \arrow[ur, "S_h"] & \widetilde{V}_h^1 \arrow[r, "d"] \arrow[ur, "S_h"] & \widetilde{V}_h^2
\end{tikzcd}$$

and equipped with canonical cochain projections Π_h^0 onto V_h^0, Π_h^1 onto V_h^1, Π_h^2 onto V_h^2, $\widetilde{\Pi}_h^0$ onto $\widetilde{V}_h^0$, $\widetilde{\Pi}_h^1$ onto $\widetilde{V}_h^1$, and $\widetilde{\Pi}_h^2$ onto $\widetilde{V}_h^2$. We set $S_{0,h} := \Pi_h^1 S_0$ and $S_{1,h} := \Pi_h^2 S_1$, assume $S_{0,h}$ is bounded uniformly in h from $\widetilde{V}_h^0$ to V_h^1, and that $S_{1,h}$ is from $\widetilde{V}_h^1$ to V_h^2. We also assume $S_{0,h}$ satisfies a surjectivity hypothesis (5.4). The method to solve (6.2) is then to find

$$\begin{gathered}
(\sigma_h, \phi_h) \in C^1([0,T], V_h^0 \times \widetilde{V}_h^0), \\
u_h \in C^1([0,T], \widetilde{V}_h^1), \\
(M_h, \rho_h) \in C^1([0,T], V_h^2 \times \widetilde{V}_h^2), \\
\lambda_h \in C^1([0,T], V_h^1),
\end{gathered}$$

such that

$$(\dot{\sigma}_h, \tau) - (u_h, d\psi) - (\lambda_h, d\tau - S_h\psi) = 0, \qquad (\tau, \psi) \in V_h^0 \times \widetilde{V}_h^0, \tag{6.3a}$$

$$(\dot{u}_h, v) + (d\phi_h, v) + (M_h, S_h v) + (\rho_h, dv) = (f, v), \qquad v \in \widetilde{V}_h^1, \tag{6.3b}$$

$$(\dot{M}_h, N) - (S_h u_h, N) = 0, \qquad N \in V_h^2, \tag{6.3c}$$

$$(\dot{\rho}_h, \mu) - (du_h, \mu) = 0, \qquad \mu \in \widetilde{V}_h^2, \tag{6.3d}$$

$$(d\sigma_h - S_h\phi_h, \zeta) = 0, \qquad \zeta \in V_h^1. \tag{6.3e}$$

We note that $d\sigma_h - S_h\phi_h = 0$ exactly if the initial conditions satisfy this condition.

Proposition 6.2. *Given initial conditions* $(\sigma_{0,h}, \phi_{0,h}) \in \Gamma_h$*,* $u_{0,h} \in \widetilde{V}_h^1$*,* $M_{0,h} \in V_h^2$*, and* $\rho_{0,h} \in \widetilde{V}_h^2$*, the system* (6.3) *has a unique solution.*

Proof. Since the system is square, we only need to show that the zero solution is the only solution whenever the initial condition, the boundary conditions, and the source term are all zero. Using an energy argument, we see that $\sigma_h = 0$, $u_h = 0$, $M_h = 0$, $\rho_h = 0$. Now, with the first equation, with $\tau = 0$, we have $(\lambda_h, S_h\psi) = 0$. Since $S_{0,h}$ is surjective, we see that $\lambda_h = 0$. This completes the proof. ∎

We set

$$(\sigma_h, \phi_h, u_h, M_h, \rho_h; \tau, v, N, \mu) := (\sigma_h, \tau) + (u_h, v) + (M_h, N) + (\rho_h, \mu),$$

and

$$\begin{aligned} &a(\sigma_h, \phi_h, u_h, M_h, \lambda_h; \tau, \psi, v, N, \zeta) \\ &\qquad := -(u_h, d\psi) + (d\phi_h, v) + (M_h, S_h v) + (\rho_h, dv) - (S_h u_h, N) - (du_h, \mu), \end{aligned}$$

and

$$b(\sigma_h, \phi_h, u_h, M_h, \lambda_h; \tau, \psi, v, N, \zeta) := (d\sigma_h - S_h\phi_h, \zeta).$$

We then define the elliptic projection

$$\begin{aligned} &(\Pi_h\sigma - \sigma, \widetilde{\Pi}_h\phi - \phi, \widetilde{\Pi}_h u - u, \Pi_h M - M, \Pi_h\rho - \rho; \tau, v, N, \mu) \\ &\ + a(\Pi_h\sigma - \sigma, \Pi_h\phi - \phi, \Pi_h u - u, \Pi_h M - M, \Pi_h\lambda - \lambda; \tau, \psi, v, N, \zeta) \\ &\ + b(\tau, v, N, \mu; \Pi_h\sigma - \sigma, \widetilde{\Pi}_h\phi - \phi, \widetilde{\Pi}_h u - u, \Pi_h M - M, \Pi_h\rho - \rho) = 0, \qquad (6.4a) \\ &b(\Pi_h\sigma - \sigma, \widetilde{\Pi}_h\phi - \phi, \widetilde{\Pi}_h u - u, \Pi_h M - M, \Pi_h\rho - \rho; \tau, v, N, \mu) = 0, \qquad (6.4b) \end{aligned}$$

for any $(\tau, \psi) \in V_h^0 \times \widetilde{V}_h^0$, $v \in \widetilde{V}_h^1$, $N \in V_h^2$, $\mu \in \widetilde{V}_h^2$, and $\zeta \in V_h^1$.

We now note the following error equations, using an argument similar to Proposition 5.6.

Proposition 6.3. *The error equations are*

$$\begin{aligned} &(\Pi_h\dot\sigma_h - \dot\sigma_h, \tau) - (\widetilde{\Pi}_h u - u_h, d\psi) - (\lambda - \lambda_h, d\tau - S_h\psi) \\ &\quad = (\Pi_h\dot\sigma - \dot\sigma, \tau) - (\widetilde{\Pi}_h u - u, d\psi) - (\lambda, (S - S_h)\psi), \\ &(\widetilde{\Pi}_h\dot u - \dot u_h, v) + (d(\widetilde{\Pi}_h\phi - \phi_h), v) + (\Pi_h M - M_h, S_h v) + (\widetilde{\Pi}_h\rho - \rho_h, dv) \\ &\quad = (\widetilde{\Pi}_h\dot u - \dot u, v) + (d(\widetilde{\Pi}_h\phi - \phi), v) + (\Pi_h M - M, S_h v) - (M, (S - S_h)v) + (\widetilde{\Pi}_h\rho - \rho, dv), \\ &(\Pi_h\dot M - \dot M_h, N) - (S_h(\widetilde{\Pi}_h u - u_h), N) \\ &\quad = (\Pi_h\dot M - \dot M, N) - (S_h(\widetilde{\Pi}_h u - u), N) + ((S - S_h)u, N), \\ &(\widetilde{\Pi}_h\dot\rho - \dot\rho_h, \mu) - (d(\widetilde{\Pi}_h u - u_h), \mu) \\ &\quad = (\widetilde{\Pi}_h\dot\rho - \dot\rho, \mu) - (d(\widetilde{\Pi}_h u - u), \mu), \end{aligned}$$

for any $(\tau, \psi) \in V_h^0 \times \widetilde{V}_h^0$, $v \in \widetilde{V}_h^1$, $N \in V_h^2$, $\mu \in \widetilde{V}_h^2$, *and* $\zeta \in V_h^1$.

6.3 Application to the Linearized EB System

We are now intersted in applying the framework just developed to the linearized EB system formulated with the complex (4.16). In this setting, we take

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathring{H}^1 \arrow[r, "\mathrm{grad}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] & L^2(\mathbb{V}) \\
0 \arrow[r, hook] & \mathring{H}^1(\mathbb{V}) \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & \mathring{H}(\mathrm{curl},\mathbb{M}) \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{vskw}"] & L^2(\mathbb{M})
\end{tikzcd}$$

along with $W^0 = L^2(\mathbb{R})$, $W^1 = L^2(\mathbb{V})$, $W^2 = L^2(\mathbb{V})$, $\widetilde{W}^0 = L^2(\mathbb{V})$, $\widetilde{W}^1 = L^2(\mathbb{M})$, and $\widetilde{W}^2 = L^2(\mathbb{M})$. Table 6.1 summarizes the spaces used. This problem can be written in the form of equations (6.2) by finding

$$\begin{aligned}
(\sigma,\phi) &\in C^0([0,T],\mathring{H}^1(\mathbb{R})\times\mathring{H}^1(\mathbb{V}))\cap C^1([0,T],L^2(\mathbb{R})\times L^2(\mathbb{V})),\\
u &\in C^0([0,T],\mathring{H}(\mathrm{curl},\mathbb{M}))\cap C^1([0,T],L^2(\mathbb{M})),\\
(M,\rho) &\in C^1([0,T],L^2(\mathbb{V})\times L^2(\mathbb{M})),\\
\lambda &\in C^0([0,T],L^2(\mathbb{V}))\cap C^1([0,T],H^{-1}(\mathrm{div},\mathbb{V})),
\end{aligned}$$

such that

$$\begin{aligned}
(\dot\sigma,\tau)-(u,\mathrm{grad}\,\psi)-(\lambda,\mathrm{grad}\,\tau - I\psi) &= 0, && (\tau,\psi)\in\mathring{H}^1\times\mathring{H}^1(\mathbb{V}),\\
(\dot u,v)+(\mathrm{grad}\,\phi,v)+(M,\mathrm{skw}\,v)+(\rho,\mathrm{curl}\,v) &= (f,v), && v\in\mathring{H}(\mathrm{curl},\mathbb{M}),\\
(\dot M,N)-(\mathrm{skw}\,u,N) &= 0, && N\in L^2(\mathbb{V}),\\
(\dot\rho,\mu)-(\mathrm{curl}\,u,\mu) &= 0, && \mu\in L^2(\mathbb{M}),\\
(\mathrm{grad}\,\sigma - I\phi,\zeta) &= 0, && \zeta\in L^2(\mathbb{V}).
\end{aligned}$$

We recall that $\overline{W}^1 = H^{-1}(\mathrm{div},\mathbb{V})$.

We choose the following finite element spaces: $V_h^0 = \mathcal{P}_1\Lambda^0$, the Lagrange $\mathcal{P}_1$ elements, $V_h^1 = \mathcal{P}_1^-\Lambda^1$, the lowest order Nédélec $H(\mathrm{curl})$ elements of the first kind, $V_h^2 = \mathcal{P}_1^-\Lambda^2$, the lowest order Raviart-Thomas elements, $\widetilde{V}_h^0 = \mathcal{P}_2\Lambda^0\otimes\mathbb{V}$, the vector-valued Lagrange $\mathcal{P}_2$ elements, $\widetilde{V}_h^1 = \mathcal{P}_2^-\Lambda^1\otimes\mathbb{V}$, the vector-valued second lowest order Nédélec $H(\mathrm{curl})$ elements of the first kind, and $\widetilde{V}_h^2 = \mathcal{P}_1\Lambda^2\otimes\mathbb{V}$, the vector-valued lowest order BDM elements. These are the same choice of spaces done in Section 5.3 for the time-independent

V	W	V_h	Variables
$\mathring{H}(\mathbb{R})$	$L^2(\mathbb{R})$	$\mathcal{P}_1\Lambda^0$	σ, τ
$\mathring{H}(\mathbb{V})$	$L^2(\mathbb{V})$	$\mathcal{P}_2\Lambda^0 \otimes \mathbb{V}$	ϕ, ψ
$\mathring{H}(\mathrm{curl}, \mathbb{V})$	$H^{-1}(\mathrm{div}, \mathbb{V})$	$\mathcal{P}_1^-\Lambda^1$	λ, μ
$\mathring{H}(\mathrm{curl}, \mathbb{M})$	$L^2(\mathbb{M})$	$\mathcal{P}_2^-\Lambda^1 \otimes \mathbb{V}$	u, v
$L^2(\mathbb{V})$	$L^2(\mathbb{V})$	$\mathcal{P}_1^-\Lambda^2$	M, N
$L^2(\mathbb{M})$	$L^2(\mathbb{M})$	$\mathcal{P}_1\Lambda^2 \otimes \mathbb{V}$	ρ, ξ

Table 6.1: Spaces used in the formulation.

EB system. Thus, we have

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathcal{P}_1\Lambda^0 \arrow[r, "\mathrm{grad}"] & \mathcal{P}_1^-\Lambda^1 \arrow[r, "\mathrm{curl}"] & \mathcal{P}_1^-\Lambda^2 \\
0 \arrow[r, hook] & \mathcal{P}_2\Lambda^0 \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "\Pi_h^1"] & \mathcal{P}_2^-\Lambda^1 \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\Pi_h^2\,\mathrm{vskw}"] & \mathcal{P}_1\Lambda^2 \otimes \mathbb{V}
\end{tikzcd}$$

and impose Dirichlet boundary conditions for the 0-forms, and null tangent component for the 1-forms. The finite elements are summarized in Table 6.1. As done in Section 5.3, we set $S_{0,h} := \Pi_h^1 S_0 = \Pi_h^1 I$ and $S_{1,h} := \Pi_h^2 S_1 = \Pi_h^2\,\mathrm{vskw}$.

The method is then to find

$$\begin{aligned}
(\sigma_h, \phi_h) &\in C^1([0,T], V_h^0 \times \widetilde{V}_h^0), \\
u_h &\in C^1([0,T], \widetilde{V}_h^1), \\
(M_h, \rho_h) &\in C^1([0,T], V_h^2 \times \widetilde{V}_h^2), \\
\lambda_h &\in C^1([0,T], V_h^1),
\end{aligned}$$

such that

$$(\dot{\sigma}_h, \tau) - (u_h, \mathrm{grad}\,\psi) - (\lambda_h, \mathrm{grad}\,\tau - S_h\psi) = 0, \qquad (\tau, \psi) \in V_h^0 \times \widetilde{V}_h^0, \tag{6.5a}$$

$$(\dot{u}_h, v) + (\mathrm{grad}\,\phi_h, v) + (M_h, S_h v) + (\rho_h, \mathrm{curl}\,v) = (f, v), \qquad v \in \widetilde{V}_h^1, \tag{6.5b}$$

$$(\dot{M}_h, N) - (S_h u_h, N) = 0, \qquad N \in V_h^2, \tag{6.5c}$$

$$(\dot{\rho}_h, \xi) - (\mathrm{curl}\,u_h, \xi) = 0, \qquad \xi \in \widetilde{V}_h^2, \tag{6.5d}$$

$$(\mathrm{grad}\,\sigma_h - S_h\phi_h, \mu) = 0, \qquad \mu \in V_h^1. \tag{6.5e}$$

The hypotheses of the abstract framework hold for this method, and so the theory developed in this chapter thus applies to this particular case.

Chapter 7

Numerical Implementation

The example we consider is gravitational waves in vacuum $f = 0$. We take the exact solution on $[0, T]$ to be

$$\boldsymbol{E} = \begin{pmatrix} -A^+ & -A^\times & 0 \\ -A^\times & A^+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \sin \omega(z - t).$$

We take $T = 7.5$, $\omega = 20$, $A^+ = 2$, and $A^\times = 1/2$, on a unit cube mesh. We use polynomials of degree $r = 1$, and, for the time integration, we use the second order Crank-Nicolson methods. The mass matrices are inverted using CG preconditioned with Jabcobi; other matrices are inverted with MINRES preconditioned with AMG.

We first implement the EB system as given by equation (4.14) with the vector de Rham complex, and present the results in Table 7.1 and Figure 7.1. We then implement a reduced version of this EB system as given in equation (4.15), and display the results in Table 7.2 and Figure 7.2. This experiment hints to the fact that, for divergence-free initial conditions, these two formulations might give equal approximations for corresponding $\boldsymbol{E}$ and $\boldsymbol{B}$.

Mesh size	Cell count	Time step	Step count	Memory
0.433013	384	0.00833333	900	945
0.216506	3072	0.00589256	1272	1357
0.108253	24576	0.00416667	1800	4613
0.0541266	196608	0.00294628	2545	30417

(a) Code information about each run

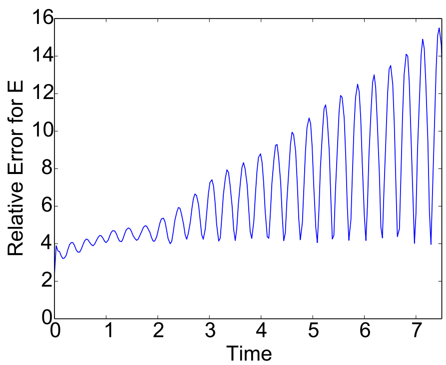
Relative Error for E	Norm of E	Absolute Error for E	Rate
375.83%	1.05E+00	3.95E+00	
144.69%	1.71E+00	2.47E+00	+0.68
53.43%	1.95E+00	1.04E+00	+1.24
13.89%	2.03E+00	2.82E-01	+1.89

(b) Errors for E

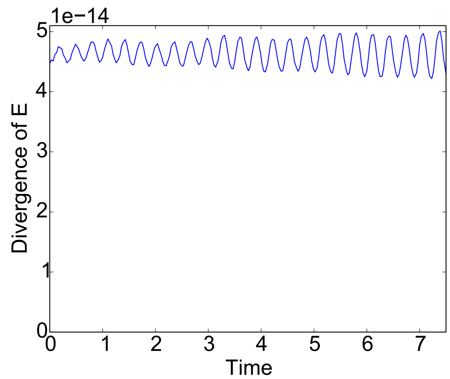
Relative Error for B	Norm of B	Absolute Error for B	Rate
438.13%	9.62E-01	4.22E+00	
78.58%	1.72E+00	1.35E+00	+1.64
18.89%	1.96E+00	3.71E-01	+1.87
8.24%	2.03E+00	1.67E-01	+1.15

(c) Errors for B

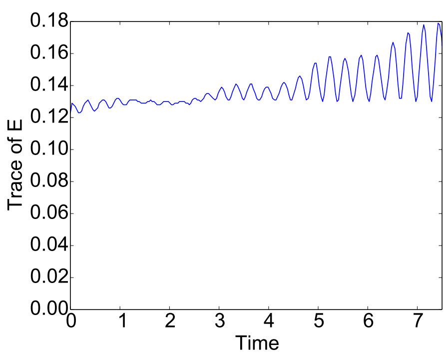
Table 7.1: Example for the EB system in 3D with Crank-Nicolson integration.



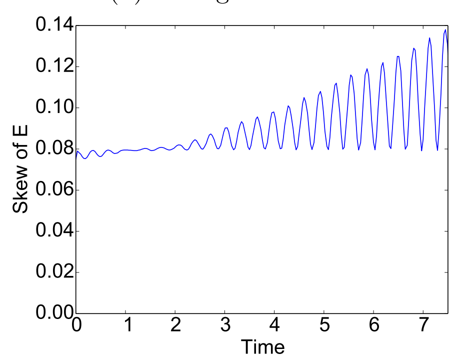
(a) Relative L^2 error for $\boldsymbol{E}$.



(b) Divergence of $\boldsymbol{E}$.

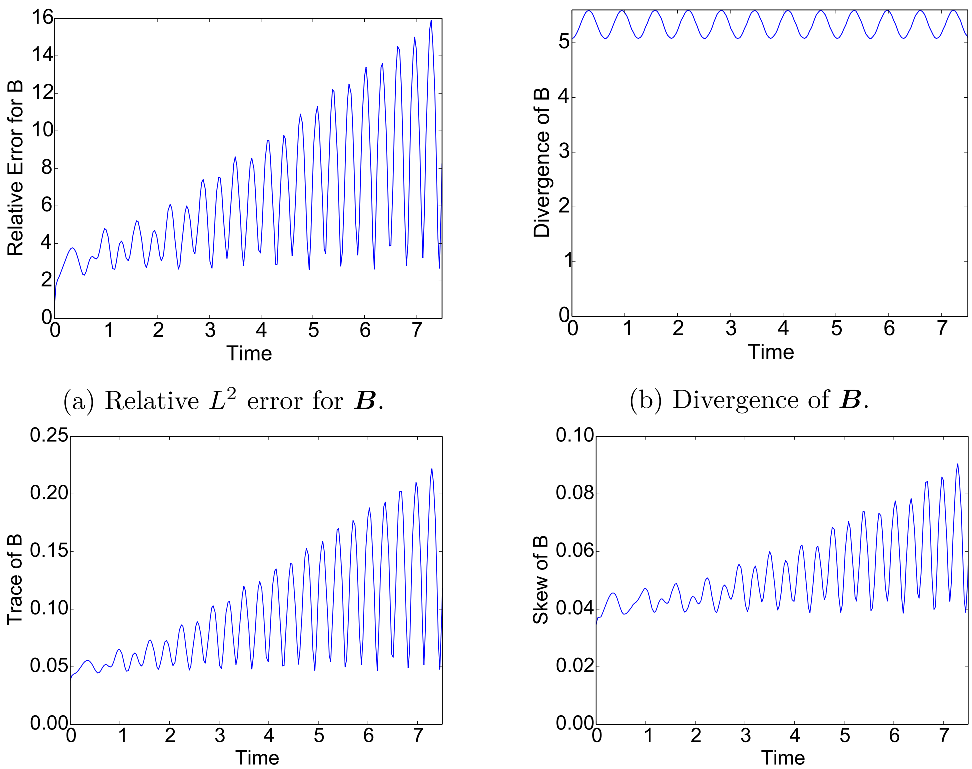


(c) Trace of $\boldsymbol{E}$.



(d) Skew part of $\boldsymbol{E}$.

Figure 7.1: Example for the EB system in 3D with Crank-Nicolson integration on mesh with 196608 cells.



(a) Relative L^2 error for $\boldsymbol{B}$.

(b) Divergence of $\boldsymbol{B}$.

(c) Trace of $\boldsymbol{B}$.

(d) Skew part of $\boldsymbol{B}$.

Figure 7.2: Example for the EB system in 3D with Crank-Nicolson integration on mesh with 196608 cells.

Mesh size	Cell count	Time step	Step count	Memory
0.433013	384	0.00833333	900	904
0.216506	3072	0.00589256	1272	1129
0.108253	24576	0.00416667	1800	2910
0.0541266	196608	0.00294628	2545	17071

(a) Code information about each run

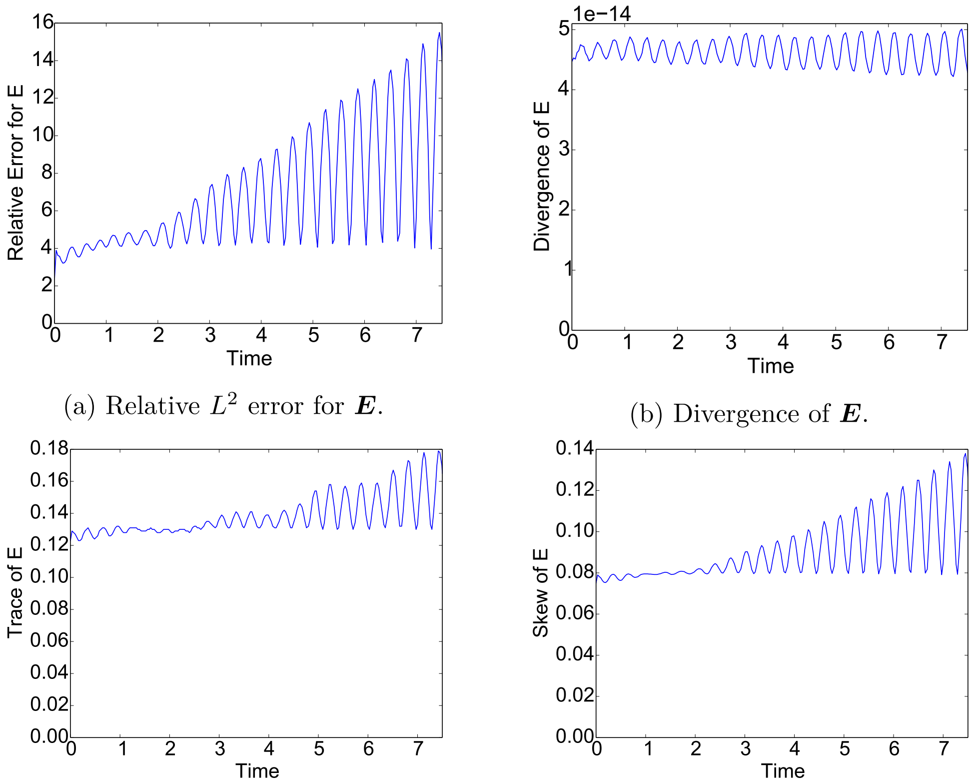
Relative Error for E	Norm of E	Absolute Error for E	Rate
375.84%	1.05E+00	3.95E+00	
144.68%	1.71E+00	2.47E+00	+0.68
53.42%	1.95E+00	1.04E+00	+1.24
13.90%	2.03E+00	2.82E-01	+1.89

(b) Errors for E

Relative Error for B	Norm of B	Absolute Error for B	Rate
438.13%	9.62E-01	4.22E+00	
78.58%	1.72E+00	1.35E+00	+1.64
18.88%	1.96E+00	3.71E-01	+1.87
8.24%	2.03E+00	1.67E-01	+1.15

(c) Errors for B

Table 7.2: Example for the reduced EB system in 3D with Crank-Nicolson integration.



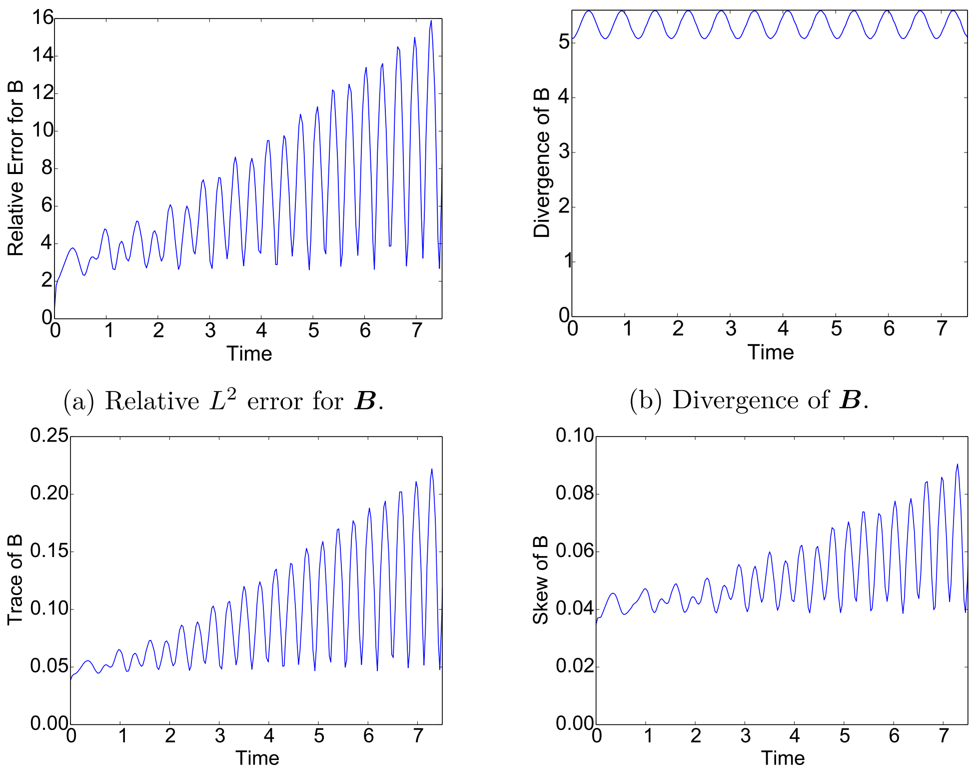
(a) Relative L^2 error for $\boldsymbol{E}$.

(b) Divergence of $\boldsymbol{E}$.

(c) Trace of $\boldsymbol{E}$.

(d) Skew part of $\boldsymbol{E}$.

Figure 7.3: Example for the EB system in 3D with Crank-Nicolson integration on mesh with 196608 cells.



(a) Relative L^2 error for $\boldsymbol{B}$.

(b) Divergence of $\boldsymbol{B}$.

(c) Trace of $\boldsymbol{B}$.

(d) Skew part of $\boldsymbol{B}$.

Figure 7.4: Example for the reduced EB system in 3D with Crank-Nicolson integration on mesh with 196608 cells.

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